APPLIED FLUID MECHANICS

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JOHN R. FREEMAN

PREFACE

For many years, the motion of fluids has been treated under various specialized headings, such as hydraulics, aeronautics, and acoustics, with the result that many teachers and students have failed to discern the relatively small number of underlying principles necessary for describing the motion of air, water, steam, oil, or any true fluid. This situation is unfortunate not only because it has unnecessarily increased the number of facts and principles that students must learn but also because it has withheld from engineers the results obtained in related fields. The rapid development of aeronautics and the application of aerodynamic principles to hydraulics and other branches of fluid mechanics have emphasized the need for a study of the principles common to all these subjects.

Fluid mechanics not only treats the external forces acting on a fluid but also recognizes the internal forces, such as those caused by viscosity, which may markedly affect the motion. The subject may be divided into hydrodynamics, which is the mathematical theory of flow, and applications of this theory. In engineering and the sciences, subjects which are wholly or in part applications of the theory of flow are aeronautics, acoustics, heat transfer, hydraulics, hydrology, lubrication, meteorology, naval architecture, dynamic oceanography, pneumatics, ventilation, and a number of others. In fact, there is hardly a phase of engineering that does not involve fluid flow in some form or other, and a knowledge of the elements of fluid mechanics has become a prerequisite to the study of any of the specialized fields mentioned.

These notes have been presented in an undergraduate course and the treatment has been fitted to the training and background of students. Although engineers may find them useful for reference, they were not prepared with such use in mind and the illustrative problems are not, in general, typical of field conditions. The authors were often tempted to include the details of interesting applications of theory, such as the hydraulic

jump and water hammer, but to do so would distract the student from the study of fundamental principles. To be sure, the hydraulic jump and water hammer are included but only as examples of certain basic relationships.

Before entering into a discussion of the principles of fluid mechanics, it will be well to recall a few definitions of terms which will be frequently used.

Fluids are defined as those substances which yield continuously to tangential forces, no matter how small. This definition includes a number of substances having high viscosity, which are commonly thought of as solids, such as glass, ice, and tar. It excludes those substances which yield only after a certain minimum stress has been exceeded, such as soap and clay. Such substances are called plastic. In general, the term *fluids* includes all substances which do not possess fixed boundaries. They are divided into two subclasses, liquids and gases, of which water and air are the outstanding examples.

Liquids are usually defined as those fluids which have a fixed volume, and this definition is sufficiently precise for practical purposes, although there are no substances possessing this attribute. Water, oil, and all other liquids are compressed slightly under pressure and in certain cases, such as the propagation of sound waves, this compressibility cannot be neglected.

Gases have neither fixed shape nor volume but tend to expand to fill any container in which they are placed. This tendency is modified by the action of gravity, which alone prevents our atmosphere from diffusing through space. However, for ordinary cases, such as the filling of an airship with helium or an engine cylinder with steam, the effect of gravity can be neglected. It is sometimes desirable to distinguish between gases and vapors. The latter possess the physical property which distinguishes a gas from a liquid but differ from a gas by being readily condensible to a liquid. Steam is the most familiar example.

Plastic substances are those which exhibit internal forces of such a nature that they will not yield continuously to tangential stresses of less than a minimum value. These stresses are usually very large in comparison with the inertia forces involved. Soap and steel are typical plastic substances, which act as solids for stresses below a critical value, but which flow when higher stresses are applied. The basic equations of motion are the

same, but simplifying assumptions are such as to give equations differing markedly from those applicable to fluids of small viscosity. For this reason they will not be included specifically in the discussion.

In the solution of problems in fluid mechanics, the fluid is assumed to be a *continuous medium.* Actually, liquids and gases are made up of molecules and atoms which are in turn composed of protons and electrons, and the actual volume occupied by these units of matter is only a small percentage of the total space that they appear to fill. However, the number of molecules in 1 cc. of gas at 0 deg. Cent. and 760 mm. pressure is 2.2×10^9 and in a liquid many times this number, so that to all appearances they are continuous and may be treated as such, provided sufficiently large quantities are considered. Many properties of fluids such as surface tension, cohesion, adhesion, and viscosity, are intimately related to their molecular structure, but in most instances these phenomena can be treated in engineering problems without recourse to a consideration of the behavior of the molecules.

A few of the physical properties of water and some other common fluids needed in the solution of the problems are given in Appendix I. For additional data the reader is referred to the Smithsonian Physical Tables and the International Critical Tables.

These notes are a gradual development from what was originally a course in hydraulics. A number of persons have contributed directly or indirectly to this development and the authors wish particularly to acknowledge their indebtedness to Dr. Baldwin M. Woods, Professor J. N. Le Conte, Dr. R. G. Folsom, and Mr. A. Ponomareff.

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CONTENTS

APPLIED FLUID MECHANICS

CHAPTER I

EQUILIBRIUM IN FLUIDS; HYDROSTATICS

1. General Assumptions.—The subject of hydrostatics is not limited to fluid systems in which there is no velocity or acceleration but rather to systems in which the fluid elements do not suffer distortion. Particles comprising the fluid maintain the same relative positions throughout the motion considered and therefore no shearing stresses arise (see Chap. IV). The velocity of fluids adjacent to fixed boundaries is normally zero and consequently hydrostatics does not include "flow" problems, although the hydrostatic equations do describe certain phases of the pressure variations occurring during flow.

It is to be noted that in those hydrostatic problems involving motion of the fluid no consideration is given to the transition period during which the pressures, velocities, and accelerations are established, because tractive forces then come into play. The significance of this statement will appear later in the treatment of specific problems.

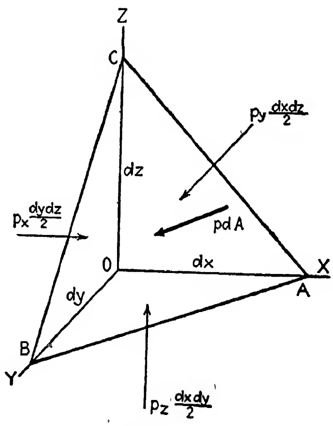

FIG. 1.

If compression is appreciable, the particular relationship between pressure and volume must be specified, thus requiring a knowledge of the process involved. The basic principles remain the same as for incompressible substances but the equa-

tions are more complex and for this reason most of the examples in this first chapter deal with liquids.

2. Derivation of General Equation.—Remembering that hydrostatics deals with fluid systems in which no tangential stresses exist, it will now be shown that the pressure, or force per unit area, at any point in the interior of a liquid is the same in all directions. Consider the elementary tetrahedron of Fig. 1 formed by a plane cutting the coordinate axes at A, B, and C, respectively, and let a unit pressure p be applied to the area dA thus cut from the plane. The force dF acting on the plane is then $p\ dA$. Suppose that the angles made by this force with the axes are α, β, and γ. The components of F parallel to the axes are

$$\left.\begin{aligned} dF_x &= p\ dA \cos \alpha \\ dF_y &= p\ dA \cos \\ dF_z &= p\ dA \cos \gamma \end{aligned}\right\}$$

Now the forces acting on the coordinate planes with intensities

$$dy\ dz$$

$$(1.2)$$

$$dF_z = p\ dx\ dy$$

but the area $dy\ dz/2 = dA \cos \alpha$, $dx\ dz/2 = dA \cos \beta$, and $dx\ dy/2 = dA \cos \gamma$. Making these substitutions in Eq. (1.2) and equating to (1.1) there result as the conditions for equilibrium

$$\begin{aligned} p\ dA \cos \alpha &= p_x\ dA \cos \alpha \\ p\ dA \cos \beta &= p_y\ dA \cos \beta \\ p\ dA \cos \gamma &= p_z\ dA \cos \gamma \end{aligned}$$

from which

$$p = p_x = p_y = p_z. \tag{1.3}$$

Equation (1.3) shows that in a fluid in hydrostatic equilibrium, the pressure is the same in all directions, since the coordinate axes were oriented arbitrarily.

The equations representing the pressure variations may be derived by considering an elementary volume of fluid of

density ρ situated at a point x,y,z (Fig. 2). The extraneous or "body" forces per unit mass, such as gravity, have components X,Y,Z parallel to the coordinate axes while the corresponding accelerations are a_x,a_y,a_z. Considering the forces parallel to the X-axis, the pressure force acting toward the right is $p\ dy\ dz$ while that acting toward the left is $-\left(p + \frac{\partial p}{\partial x}dx\right)dy\ dz$. (Forces are considered positive when acting in the positive direction of the coordinates.) The equilibrium equation for the X-direction is then

$$\Sigma F_x = 0 = p\ dy\ dz - \left(p+\frac{\partial p}{\partial x}dx\right)dy\ dz - a_x\rho\ dx\ dy\ dz + X\rho\ dx\ dy\ dz$$

Dividing by $dx\ dy\ dz$ and simplifying gives

$$\frac{\partial p}{\partial x} = \rho(X - a_x) \qquad (1.4a)$$

Similarly for the forces parallel to the other axes,

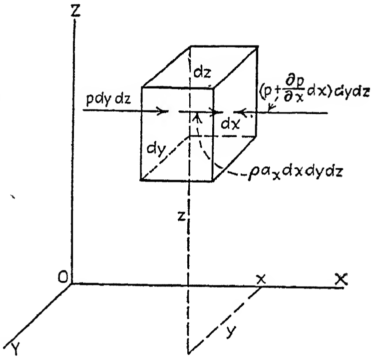

FIG. 2.

$$\frac{\partial p}{\partial y} = \rho(Y - a_y) \qquad (1.4b)$$

$$\frac{\partial p}{\partial z} = \rho(Z - a_z) \qquad (1.4c)$$

If the pressure at x,y,z is p, the differential is

$$dp = \left(\frac{\partial p}{\partial x}\right)_{yz} dx + \left(\frac{\partial p}{\partial y}\right)_{xz} dy + \left(\frac{\partial p}{\partial z}\right)_{xy} dz$$

or

$$dp = \rho[(X - a_x)dx + (Y - a_y)dy + (Z - a_z)dz] \qquad (1.5)$$

which is the differential equation of equilibrium in fluids.

The remainder of this chapter will be devoted to a number of problems of equilibrium, all of which may be solved by application of Eq. (1.5) and the principles of mechanics.

3. Surface of Liquid in a Rotating Vessel.—If water be placed in a cylindrical vessel which is rotated about its vertical axis, the liquid finally comes to a state of equilibrium in which all of

its particles rotate with the same angular velocity as the vessel. The variation of pressure throughout the liquid and the shape of the surface curve may be determined.

If a section through the axis of the vessel be considered, the problem becomes two-dimensional (z and r) since the vessel is symmetrical about its axis. In Eq. (1.5) it is necessary to evaluate the inertia forces per unit mass, a_r and a_z. It will be convenient to take the origin of coordinates on the axis at the bottom of the vessel and make the outward and upward directions positive. Now consider the particle of liquid whose coordinates are z and r (Fig. 3). The inertia force along the R-axis (away from the center) is

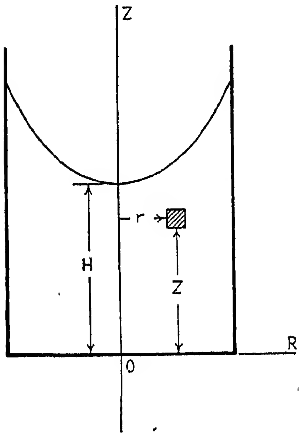

FIG. 3.

$$\rho a_r = -\omega^2 \rho r$$

The extraneous force per unit mass due to gravity is $Z = -g$. Equation (1.5) becomes

$$dp = \qquad dr - g\,dz)$$

Integrating,

$$p = \rho\frac{\omega^2 r^2}{2} - \rho g z + C.$$

C may be evaluated by the conditions at the surface of the liquid. If $z = H$ where $r = 0$ and $p = 0$, the last equation becomes

$$0 = 0 - \rho g H + C$$

or

$$C = \rho g H$$

Then the pressure at any point in the vessel is given by the expression

$$p = \rho\left(\frac{\omega^2 r^2}{2} -\right.$$

In order to find the shape of the surface curve, note that at every point on the surface, $p = 0$. Then

$$0 = \rho\left(\frac{\omega^2 r^2}{2} - gz + gH\right)$$

from which

$$z = H + \frac{\omega^2 r^2}{2g}$$

4. Manometers.—A manometer is a device for measuring pressure by balancing the pressure against a column of liquid in static equilibrium. Manometers may be vertical or inclined.

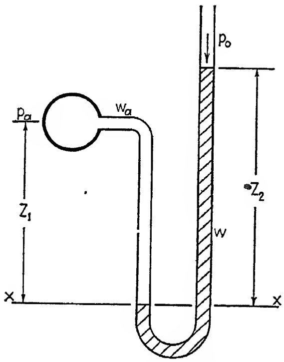

FIG. 4.—Open manometer.

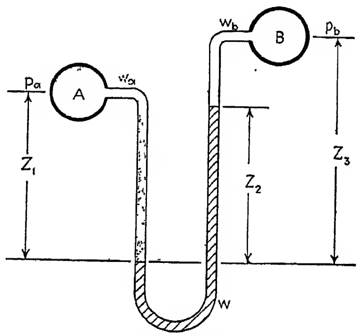

FIG. 5.—Differential manometer.

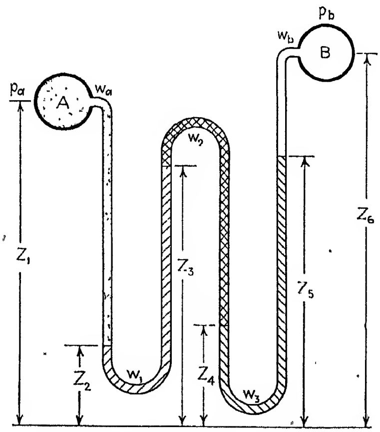

FIG. 6.—Compound manometer.

"open" (Fig. 4), "differential" (Fig. 5), or "compound" (Fig. 6). Liquids are generally used for manometer fluids as their density remains nearly constant under all practical working pressures, thus simplifying the computations.

The basic equation for computing the pressures indicated by manometers is Eq. (1.5). Since gravity is the only force acting, it reduces to

$$dp = -\rho g\, dz \tag{1.6}$$

z being positive when measured upward. Integrating, and letting $p = p_0$ where $z = z_0$,

$$p = \rho g(z_0 - z) + p_0$$

Taking the origin at the surface of the fluid, and noting that $\rho g = w$,

$$p = p_0 - wz \tag{1.7}$$

In the solution of manometer problems it will be found convenient to write this equation as

$$p = p_0 + wz \tag{1.8}$$

using for z its absolute value instead of the negative value obtained by measuring down from the surface.

In general it is best not to try to apply a formula to manometer installations but to work out each one as a separate problem. A few examples will be given.

1. *The Open Manometer* (Fig. 4).—It is evident that the pressure in each leg of the manometer at the level $x - x$ is the same. Applying Eq. (1.8) to the right leg,

$$p_x = p_0 + z_2 w$$

and to the left leg,

$$p_x = p_a + z_1 w_a$$

Equating and solving for p_a,

$$p_a = p_0 + z_2 w - z_1 w_a \tag{1.9}$$

p_0 is usually atmospheric pressure.

It is worth noting at this point that the units of p_a will depend entirely on the units of the remaining quantities of the equation, and that these units must be consistent. For example, with w in pounds per cubic foot and z in feet, p_0 and p_a must be expressed in pounds per square foot. The matter of dimensional homogeneity will be taken up in more detail in a later chapter.

2. *The Differential Manometer* (Fig. 5).—The differential manometer, represented in Fig. 5, is used to measure the pressure

difference existing between points A and B. The principle of solution is the same as for the open manometer. For the right leg,

$$p_x = p_b + (z_3 - z_2)w_b + z_2 w$$

and for the left leg,

$$p_x = p_a + z_2 w_a + (z_1 - z_2)w_a$$

Equating,

$$p_b + (z_3 - z_2)w_b + z_2 w = p_a + z_2 w_a + (z_1 - z_2)w_a$$

from which

$$p_a - p_b = (z_3 - z_2)w_b - (z_1 - z_2)w_a + z_2(w - w_a)$$

If the fluids at A and B have the same density, as is frequently the case,

$$p_a - p_b = z_2(w - w_a) - (z_1 - z_3)w_a$$

or

$$\left(\frac{p_a}{w_a} + z_1\right) - \left(\frac{p_b}{w_a} + z_3\right) = z_2\left(\frac{w - w_a}{w_a}\right) \qquad (1.10)$$

Equation (1.10) is the form most generally used. The deflection of a differential manometer is seen to be a measure of the drop in the hydraulic grade line. (See Chap. III for a discussion of the hydraulic grade line.)

When the fluids at A and B are gases, their density is frequently so low in comparison with the liquid used in the manometer that the force due to the weight of the gas may be neglected for practical applications.

The solution of the compound manometer will not be taken up as it requires no new principle, being only more involved. It will be left as an exercise for the student.

5. Center of Pressure.—It has been shown that the pressure in a liquid acted upon by gravity alone is

$$p_0 - wz \qquad (1.7)$$

due attention being paid to the sign of z. In a great many hydraulic computations atmospheric pressure is assumed as the

datum, making $p_0 = 0$. Thus is obtained the pressure due to the liquid alone.

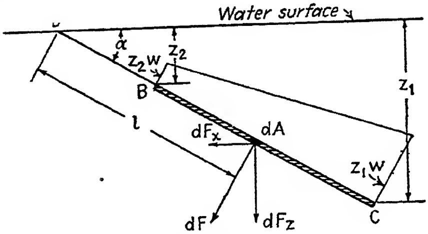

FIG. 7.

Considering Fig. 7, BC represents a plate submerged in a liquid whose surface is at D. The force exerted on an element dA of the plate is

$$dF = -wz\,dA.$$

The horizontal and vertical components of this force are

$$dF_x = -wz \sin \alpha\, dA$$
$$dF_z = -wz \cos \alpha\, dA$$

Since $z = l \sin \alpha$,

$$dF = -wl \sin \alpha\, dA$$

Integrating,

$$F = -w \sin \alpha \int_0^A l\, dA$$

But $\int_0^A l\, dA$ is the moment of the area about D and is equal to L_cA where L_c is the distance from D to the center of gravity of the area.

Hence

$$F = -w \sin \alpha L_c A$$

or

$$F = -wZ_cA$$

where Z_c is the vertical distance from the surface of the liquid to the center of gravity of the area. Similarly

$$F_x = -wZ_cA \sin \alpha$$

and

$$F_z = -wZ_cA \cos \alpha$$

Now if the center of pressure of the plate BC is defined as that point at which a single force of magnitude F, but opposite in direction, should be applied in order to hold the plate in statical equilibrium, the location of this point may be found by summing up the forces and moments. Writing moments about D,

$$dM = l\,dF$$

But

$$dF = -wz\,dA, \qquad z = l \sin \alpha$$

hence,

$$dM = -wl^2\,dA \sin \alpha$$

Integrating,

$$M = -w \sin \alpha \int_0^A l^2\,dA$$

Here, $\int_0^A l^2\,dA$ is the moment of inertia of the area A about an axis through D. Calling this I_0,

$$M = -w \sin \alpha I_0$$

The force acting is

$$F = -wZ_cA$$

By definition the moment arm of this force necessary to maintain equilibrium is L_{cp} where L_{cp} is the distance along the plate from the surface to the center of pressure. Then

$$M = FL_{cp}$$

or

$$L_{cp} = \frac{M}{F}$$

Substituting for M and F their values from immediately above,

$$L_{cp} = \frac{-w \sin \alpha I_0}{-wZ_cA}$$

Noting that

$$Z_{cp} = L_{cp} \sin \alpha$$
$$Z_c = L_c \sin \alpha$$
$$I_0 = \int_0^A l^2\,dA$$

and

$$L_cA = \int_0^A l\,dA$$

the vertical distance to the center of pressure is given by the expression

$$Z_{cp} = \frac{\sin \alpha \int_0^A l^2 \, dA}{\int_0^A l \, dA} \tag{1.11}$$

6. Stability of Dams.—A gravity dam is in equilibrium under the action of the hydrostatic forces, its own weight, and the resistance of the foundations. The pressure on the foundation can be determined from the distribution of the weight and the hydrostatic forces.

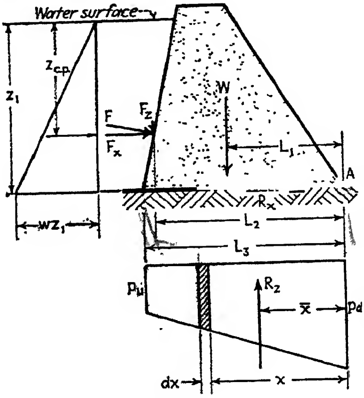

FIG. 8.

Figure 8 represents a section of a trapezoidal dam. Its weight is W, and its center of gravity is a distance L_1 from the toe of the structure at A. The water behind it has a depth of z_1 and exerts a force F on the face of the dam a distance L_2 from the toe. The length of the base is L_3.. The horizontal force which resists the sliding of the dam on its foundation is R_x and the vertical reaction on the foundation is R_z passing through the base a distance $\bar{x}$ from the toe. The unit stresses at the heel and toe of the dam are p_u and p_d, respectively.

The conditions for equilibrium are

$$\Sigma F_x = 0$$
$$\Sigma F_z = 0$$
$$\Sigma M = 0$$

Considering unit length perpendicular to the plane of the figure, the equilibrium equations are

$$\Sigma F_x: \qquad F_x - R_x = 0$$
$$\Sigma F_z: \qquad W + F_z - R_z = 0$$
$$\Sigma M_A: F_z L_2 + W L_1 - F_x(z_1 - z_{cp}) - R_z \bar{x} = 0$$

The values of R_x, R_z, and $\bar{x}$ are obtained directly by solution of these equations:

$$R_x = F_x$$
$$R_z = W + F_z$$
$$\bar{x} = \frac{F_z L_2 + W L_1 - F_x(z_1 - z_{cp})}{R_z}$$

The unit pressures p_u and p_d at heel and toe, respectively, may be obtained as follows, assuming a linear distribution of pressure along the base: the vertical reaction is

$$R_z = \left(\frac{p_u + p_d}{2}\right)L_3$$

and the total moment of the upward forces about the toe at A is

$$R_z\bar{x} = \int_0^{L_3} px\,dx = L_3^2\left(\frac{p_d}{6} + \frac{p_u}{3}\right)$$

Solving these equations simultaneously for p_d and p_u, and rewriting the last two equations,

$$R_z = \frac{p_d L_3}{2} + \frac{p_u L_3}{2} \tag{1.12}$$

$$R_z\bar{x} = \frac{p_d L_3^2}{6} + \frac{p_u L_3^2}{3} \tag{1.13}$$

Multiplying Eq. (1.12) by L_3, and Eq. (1.13) by 3

$$R_z L_3 = \frac{p_d L_3^2}{2} + \frac{p_u L_3^2}{2} \tag{1.14}$$

$$3R_z\bar{x} = \frac{p_d L_3^2}{2} + p_u L_3^2 \tag{1.15}$$

Subtracting Eq. (1.14) from Eq. (1.15),

$$R_z(3\bar{x} - L_3) = p_u\frac{L_3^2}{2}$$

Solving for p_u,

$$p_u = \frac{2R_z(3\bar{x} - L_3)}{L_3^2}$$
$$= \frac{6R_z\bar{x}}{L_3^2} - \frac{2R_z}{L_3} \tag{1.16}$$

Multiplying Eq. (1.12) by $2L_3$ and subtracting Eq. (1.15) from the product leaves

$$R_z(2L_3 - 3\bar{x}) =$$

Solving for p_d,

$$= \frac{2R_z}{L_3} - p_u \tag{1.17}$$

Since masonry can withstand practically no stress in tension, it is necessary that p_u and p_d both be positive, or compressive, stresses. It will be seen by inspection of Eqs. (1.16) and (1.17) that for $\bar{x} < L_3/3$, p_u is negative and for $\bar{x} > 2L_3/3$, p_d is negative. Thus, it is seen that in order to have no tension in the base, the resultant of the horizontal and vertical forces must fall within the middle third.

For stability the dam must be able to resist overturning and sliding. For overturning, the condition is that the resultant fall within the base. This condition is already satisfied if no tension exists. For sliding, the condition to be satisfied is that

where f is the coefficient of friction on the base.

In most practical cases, water seeps under the dam at the foundation and exerts an upward pressure which tends to reduce the stability of the structure. The amount of this upward pressure will not be discussed here, but it will be noted that the mere presence of an uplift pressure in itself has no special significance if all the criteria of stability are satisfied when the uplift forces are considered.

The discussion has been confined to the base of the dam and its contact with the foundation. The same considerations apply to every horizontal section. A complete analysis would begin at the top of the dam and investigate its stability at regular intervals from top to bottom.

7. Floating Bodies.—The general equation of static equilibrium leads to Archimedes' principle that a body immersed in a liquid apparently loses weight in amount equal to the weight of liquid displaced. A corollary to this principle is that a body floating at the surface of separation of two fluids displaces a weight of the lower fluid equal to its own weight minus the weight of the upper

fluid displaced. This general phenomenon is known as *buoyancy*. On the surface of lakes and seas, the weight of air displaced by a floating body is small and the weight of the water displaced is said to be equal to the weight of the object.

Consider an elementary cube of density ρ_2 immersed in a liquid of density ρ_1 and with its faces parallel to the coordinate planes (Fig. 9). Going back to Eq. (1.5) and noting now that gravity is the only force acting,

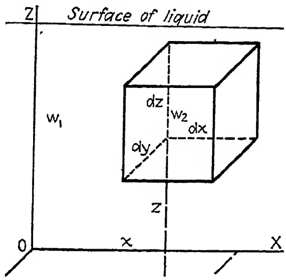

FIG. 9.

$$dp = -\rho g\, dz \qquad (1.6)$$

or

$$\begin{aligned} p &= -\rho g z + C \\ &= -wz + C \end{aligned}$$

The forces acting on the cube are

$$\begin{aligned} &\text{Pressure on the bottom} &&= (-w_1 z + C)dx\, dy \\ &\text{Pressure on the top} &&= -[-w_1(z + dz) + C]dx\, dy \\ &\text{Weight of the cube} &&= -w_2\, dx\, dy\, dz \end{aligned}$$

and the pressures on the sides which have no vertical components. The sum of the forces is

$$dR = -w_1 z\, dx\, dy + C\, dx\, dy + w_1 z\, dx\, dy + w_1\, dz\, dx\, dy - C\, dx\, dy - w_2\, dx\, dy\, dz$$

which reduces to

$$dR = (w_1 - w_2)dx\, dy\, dz$$

For a liquid or a small change of elevation in a gas, w_1 is nearly constant and

$$R = (w_1 - w_2)\int\int\int dx\, dy\, dz$$

But $\int\int\int dx\, dy\, dz$ is the volume of the body, and hence

$$R = (w_1 - w_2) \times \text{Volume} \qquad (1.18)$$

The body sinks if $w_2 > w_1$ and rises if $w_2 < w_1$.

The resultant force due to buoyancy acts through the center of gravity of the displaced fluid. This center of gravity is called the *center of buoyancy* and, when a floating body is in statical equilibrium, is in the same vertical line as the center of gravity of the body.

The stability of a body in floating equilibrium depends on the moments set up when the body is slightly displaced. The resulting moments may tend to restore the body to its original position, to move it still farther from that position, or to maintain it in equilibrium. The equilibrium is said in each case to be stable, unstable, or neutral, respectively.

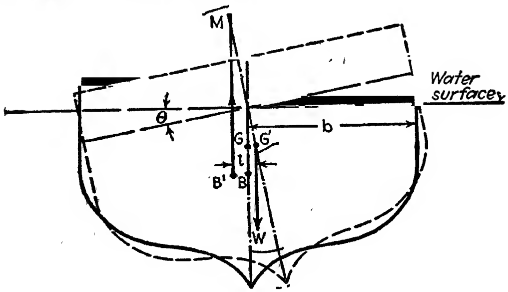

FIG. 10.

Consider a cross section of a ship (Fig. 10) which is symmetrical about a vertical longitudinal plane. G is the center of gravity and B the center of buoyancy when in equilibrium. G' and B' are the centers of gravity and buoyancy for the displacement shown. The point where the line of action of the buoyant force intersects the line $G'M$ is called the *metacenter*, and its position determines the stability of the ship. If the metacenter is above the center of gravity, the moment set up will tend to restore the ship to stable equilibrium. The amount of the restoring moment is the product of the weight of the vessel and the horizontal movement of the center of buoyancy relative to the center of gravity.

8. Rectilinear Acceleration.—An oil barrel located on an elevator or the fuel tanks of an airplane show pressure variations during accelerations, which differ from those resulting from the weight of the liquid alone. Problems of this type can be solved more or less mechanically by the application of Eqs. (1.4) and (1.5). Two examples will illustrate the method.

1. Pressure variation in a liquid during vertical acceleration.

$$\text{Extraneous force per unit mass} = -g$$
$$\text{Acceleration upward} = +a_z$$
$$\Delta p = -\rho(g + a_z)\Delta z$$

2. Pressure variation in a liquid during horizontal acceleration.

$$\text{Extraneous force per unit mass} = -g$$
$$\text{Horizontal acceleration} = +a_x$$
$$dp = -\rho(a_x\, dx + g\, dz)$$

The lines of constant pressure correspond to $dp = 0$ and are inclined at the angle

$$\frac{dz}{dx} = -\frac{a_x}{g}$$

This is also the angle of inclination of the free surface.

9. Variation of Pressure in the Atmosphere.—To illustrate the application of the hydrostatic equations to compressible fluids, they will now be used to compute the vertical pressure gradient in the earth's atmosphere. The conditions assumed are seldom found in nature and for a detailed discussion of this problem students are referred to standard works on meteorology. The general equation of equilibrium in a static fluid under the action of gravity is

$$dp = -\rho g\, dz \tag{1.6}$$

or

$$dz = -\frac{dp}{\rho g} = -\frac{dp}{w}$$

In the atmosphere, both p and ρ are variable and a relation must be established between them before the integration can be carried out.

For perfect gases and approximately for air,

$$pv = RT \tag{1.19}$$

where p = the pressure, pounds per square foot.
v = the volume of 1 lb. of gas.
R = the gas constant.
T = the absolute temperature, degrees Fahrenheit + 460.

Three general types of change of state possible for a gas are:
1. Constant volume ($p = KT$),
2. Constant temperature ($pv = K'$),
3. Constant heat (adiabatic) ($pv^n = K''$),

where $n = c_p/c_v$. c_p and c_v are the specific heats at constant pressure and constant volume, respectively. Appendix I gives values of these constants for air and a few common gases.

The adiabatic condition applies approximately to a dry atmosphere and will be assumed in the following discussion. By definition

$$v = \frac{1}{w}$$

and

$$dz = -v\,dp$$

Substituting from Eq. (1.19),

$$dz = -\frac{RT}{p}dp \tag{1.20}$$

For an adiabatic change of state,

$$pv^n = p_1 v_1{}^n$$

and

$$p = p_1\left(\frac{T_1}{T}\right)^{\frac{n}{1-n}} = p_1 T_1{}^{\frac{n}{1-n}} T^{\frac{-n}{1-n}} \tag{1.21}$$

Differentiating with respect to T,

$$dp = -\frac{n}{1-n}p_1 T_1{}^{\frac{n}{1-n}} T^{\frac{-1}{1-n}} dT \tag{1.22}$$

Substituting Eqs. (1.21) and (1.22) in Eq. (1.20),

$$dz = -\frac{RT\left(-\frac{n}{1-n}p_1 T_1{}^{\frac{n}{1-n}} T^{\frac{-1}{1-n}}\right)}{p_1 T_1{}^{\frac{n}{1-n}} T^{\frac{-n}{1-n}}} dT$$

Reducing,

$$dz = \frac{n}{1-n}R\,dT \tag{1.23}$$

Integrating,

$$z = \frac{n}{1-n}RT + C$$

To evaluate C, let $T = T_0$ when $z = 0$.

$$0 = \frac{\quad}{1 - n} RT_0 + C \qquad \text{and} \qquad C \qquad 1 -$$

Then

$$z = \frac{n}{1 - n} RT - \frac{n}{1 - n} RT_0$$

and

$$T = \frac{z + \dfrac{n}{1 - n} RT_0}{\dfrac{n}{1 - n} R}$$

$$T = T_0 + \frac{z(1 - n)}{nR} \tag{1.24}$$

The temperature decrease in an adiabatic dry atmosphere for a decrease in elevation of 100 ft. is found by a rearrangement of Eq. (1.23).

$$\Delta T = \frac{1 - n}{nR} \Delta z$$

Substituting the numerical values,

$$\Delta T = \frac{1 - 1.40}{1.40 \times 53.3} \times 100 = -0.54 \text{ deg. Fahr. (approx.)}$$

Similarly, if the temperature at a given elevation is known, it can be found for other elevations by direct application of Eq. (1.24).

The variation in pressure can be obtained by substituting the value of T from Eq. (1.24) into Eq. (1.20):

$$\frac{dp}{p} = -\frac{dz}{\left[RT_0 - \dfrac{n - 1}{n} z\right]}$$

Reducing this expression and integrating between the limits $z = 0$ to $z = z$, and $p = p_0$ to $p = p$,

$$p = p_0 \left[1 - \frac{(n - 1)z}{nRT_0}\right]^{\frac{n}{n - 1}} \tag{1.25}$$

At an elevation of 16,000 ft. and a surface temperature of 60 deg. Fahr. (520 deg. abs.), the pressure is approximately $0.4p_0$. This relation can not be applied to greater elevations as the atmosphere tends to become isothermal in the upper regions and the composition differs markedly from that at the earth's surface.

10. Hydraulic Press.—So many devices in everyday use involve the principles of hydrostatics that no attempt will be made to present a complete survey of such applications. However, the hydraulic press or hydraulic jack is of such universal usage as to deserve mention. Referring to Fig. 11, a force W_1

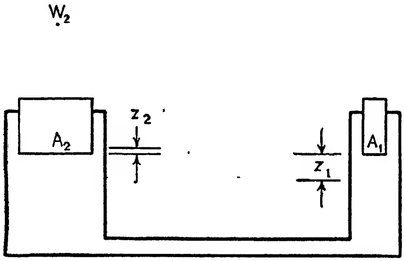

FIG. 11.

is applied to a piston of area A_1 and causes motion or holds in position a piston of area A_2 carrying a load W_2. Friction will be neglected and the motion, if any, will be assumed to be so slow that inertia effects are negligible as compared with the hydrostatic forces.

If the pistons are initially at the same elevation, they will move during some interval of time through distances z_1 and z_2, respectively. From the standpoint of pressures,

$$p_1 = p_2 + (z_1 + z_2)w$$
$$W_2 = p_2A_2 = W_1\frac{A_2}{A_1} - A_2w(z_1 + z_2)$$

The respective displacements are obtained from a consideration of the displacement volumes as $A_2z_2 = A_1z_1$.

From the standpoint of work and change of energy,

$$W_1\,dz_1 + wA_1\,dz_1 = W_2\,dz_2 + wA_2\,dz_2$$
$$A_1\,dz_1 = A_2\,dz_2$$

Therefore,

$$W_2 = W_1\frac{A_2}{A_1} - A_2w(z_1 + z_2)$$

The principle involved is analogous to that of the lever, the ratio of the piston areas replacing the ratio of lever arms, and the hydrostatic pressure difference corresponding to the weight of the lever arms.

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Problems

Data for problems dealing with the physical properties of fluids are to be found in Appendix I.

1. If water is near the freezing point, find the percentage change in volume when the pressure increases from 14.7 to 200 lb. per sq. in. Determine the corresponding volume change with oxygen if the temperature is maintained at 32 deg. Fahr.

Ans. (*a*) 0.0687 per cent decrease; (*b*) 92.7 per cent decrease.

2. What will be the weight per unit volume of water if it is subjected to a pressure of 2000 atm.? Assume temperature constant at 0 deg. Cent.

Ans. Approximately 67.8 lb. per cu. ft.

3. In expressing 100.00 lb. of water in cubic feet, what is the allowable variation in temperature measurement in order to obtain an answer within

0.1 per cent? Assume the temperature to be 68 deg. Fahr.

Ans. About 8 deg.

4. Consult standard physical tables or handbooks, and report on the capillary rise of water in a glass tube of $\frac{1}{8}$ in. internal diameter if the air and water temperature is 68 deg. Fahr. *Ans.* 0.368 in.

5. A steel penstock of 30 in. inside diameter is exposed to a head of 500 ft. of water. What should be the minimum thickness of wall for an allowable stress of 9000 lb. per sq. in.? *Ans.* 0.361 in.

6. A mercury manometer is to be used to measure the pressure of a column of water. The water column may reach a height of 50 ft. How long must the mercury column be to balance it? *Ans.* 3.69 ft.

7. The depth of water in a flume is measured by a gage in a stilling well connected to the bottom of the flume by a small pipe. The water in the flume is 7 ft. deep. If the water in the flume is at 45 deg. Fahr. and that in the pipe is at 70 deg. Fahr., what is the error in the gage reading?

Ans. 0.015 ft.

8. A closed cylindrical tank 4 ft. in diameter and 5 ft. deep is filled with oil having a sp. gr. of 0.88. If the tank is rotated about its axis at a speed of 250 r.p.m. what is the unit pressure (pounds per square inch) at the bottom of the vertical side? *Ans.* 18.1 lb. per sq. in.

9. An open cubical container, 2 ft. on a side, is filled with water to a height of 1.6 ft. Find the necessary acceleration in the horizontal direction in order to spill water over the rear edge of the container.

Ans. 12.9 ft. per sec.^{2}

10. A cylinder is 1 ft. in diameter and is filled with water to a depth of 3 ft. Write the equation for the water surface if the cylinder is rotated at 300 r.p.m. and determine the maximum elevation above the original level.

11. A barrel of oil 4 ft. high is to be moved upward in an elevator which has an upward acceleration of 5 ft. per sec.^{2} at starting and -10 ft. per sec.^{2} at stopping. Compute the pressure difference between the top and bottom of the barrel under these conditions. Assume a reasonable value for the specific gravity of oil.

12. The differential manometer of Fig. 5 contains CCl_4 (sp. gr. 1.596) and water. If z_1 is 5.00 ft., z_2 is 2.36 ft., and z_3 6.50 ft., what is the difference in pressure between the two pipes, expressed in pounds per square inch?

13. Imagine the compound manometer of Fig. 6 to be open at the right end. The fluid w_1 is mercury having a sp. gr. of 13.6, and w_3 is CCl_4 having a sp. gr. of 1.600. The pipe and all connecting lines are filled with water except for w_5 which of course is air. z_1 is 5.00 ft., z_2 and z_4 are 6 in., and z_3 and z_5 are 2.50 ft. What is the pressure in the pipeline in pounds per square inch? *Ans.* 10.5 lb. per sq. in.

14. A U tube containing carbon tetrachloride and water reaches equilibrium at conditions shown in Fig. 12. What is the specific gravity of the carbon tetrachloride?

15. An inclined draft gage filled with oil of sp. gr. 0.85 indicates as shown in Fig. 13 when connected to a boiler firebox. If the barometer reads 29.87 in. of mercury, find the absolute pressure in the firebox.

16. Connections are made to the top and bottom of a pipe in the same plane perpendicular to its longitudinal axis. A differential manometer between the two connections shows a deflection d with a manometer fluid of sp. gr. s. The pipe and manometer connections are filled with water. What does the manometer deflection measure?

17. A funnel, in the shape of a cone having a base area of 1 sq. ft. and an altitude of 1 ft., has a small hole at the vertex. If the base rests on the smooth surface of a platform scale and in this inverted position the funnel is filled with water, what downward pressure must be exerted upon it to maintain equilibrium and prevent the escape of water between the base of the funnel and the platform? Neglect the weight of the funnel. What is the least scale reading possible under these conditions?

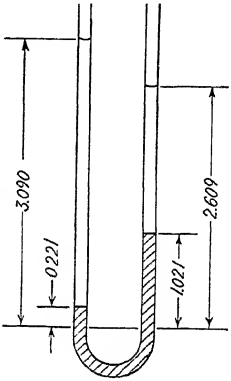

Fig. 12.

18. A rectangular gate 6 ft. high and 5 ft. wide is placed vertically with its upper edge 4 ft. below a free water surface. What force must be applied at the bottom of the gate to keep it closed if it is hinged at the upper edge? What is the depth to the center of pressure? *Ans.* (*a*) 7490 lb. (*b*) 7.43 ft.

19. The gate of Prob. 18 is placed so that the water surface is 3 ft. above its upper edge on one side. The water surface on the other side is 2 ft. lower. What force must be applied at its lower edge to keep it closed?

20. A plane T-shaped surface has dimensions as follows: The stem is 6 ft. wide and 8 ft. high; the crosspiece is 4 ft. high and 10 ft. long. This surface is submerged in water with the end of the stem at the water surface and making an angle of 30 deg. with the horizontal. Find the depth to the center of pressure and the resultant force on one side.

21. A dam of concrete weighing 150 lb. per cu. ft. is triangular in section with its vertical face upstream. The height is 40 ft. and the base length 30 ft. If water stands against it to a depth of 36 ft., compute the following: (*a*) the total horizontal thrust per foot of width; (*b*) the location of the center of pressure, (*c*) the coefficient of friction necessary to prevent sliding.

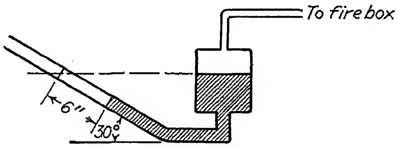

Fig. 13.

22. A concrete dam has dimensions as shown in Fig. 14. Assuming the unit weight of concrete as 150 lb. per cu. ft., find the unit stress at the upstream and downstream edges of the dam. If tensile stress occurs at the heel, how much must the base be extended downstream?

23. A vertical gate in the side of a water tank is 3 ft. wide and 4 ft. high, with its upper edge 6 ft. below the water surface. Assuming that the gate is hinged at the bottom and rests against a stop at the top, find (*a*) the depth

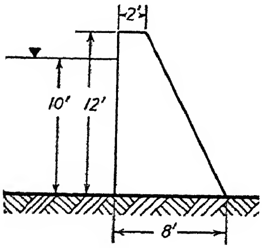

FIG. 14.

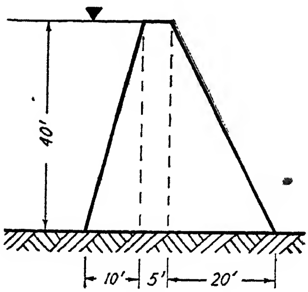

FIG. 15.

to the center of pressure on the gate, (*b*) the total force acting on the gate, (*c*) the reaction on the hinge, (*d*) the reaction on the stop.

Ans. (*a*) 8.16 ft. below surface, (*b*) 5990 lb.; (*c*) 3240 lb.; (*d*) 2750 lb.

24. In Prob. 23, assume that the fluid is oil of sp. gr. 0.92 instead of water and that oil rises on the outside of the tank to the top of the gate. Compute the same quantities as in Prob. 23.

25. A gate in the side of a tank is 5 ft. wide and 10 ft. long. The 5-ft. edge is horizontal and the gate makes an angle with the horizontal of 30 deg. The water surface is 10 ft. above the lower edge. Find (*a*) the total force acting on the gate, (*b*) the horizontal component, (*c*) the vertical component, (*d*) the vertical distance from the water surface to the center of pressure.

Ans. (*a*) 23,400 lb.; (*b*) 11,700 lb.; (*c*) 20,250 lb.; (*d*) 7.77 ft. below surface.

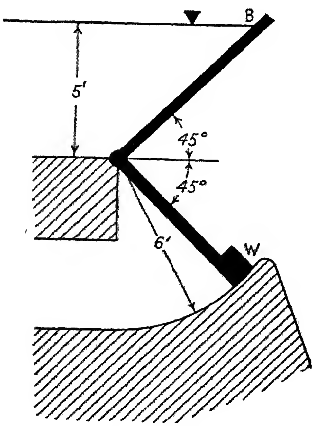

FIG. 16.

26. The lower corner of a water tank has the shape of a quadrant of a circle with a 3-ft. radius. The water surface is 4 ft. above the center of curvature. Considering a section 4 ft. long, what are (*a*) the horizontal and (*b*) vertical components of pressure and (*c*) magnitude and (*d*) direction of the resultant force?

Ans. (*a*) 4110 lb.; (*b*) 4755 lb.; (*c*) 6295 lb.; (*d*) 40° 52′ with vertical.

27. If the weight of material in the dam shown in Fig. 15 is 150 lb. per cu. ft. and the coefficient of sliding friction is 0.50, find unit stresses at toe and heel and determine if the dam is safe against overturning and sliding.

28. Neglecting the weight of the arms of the automatic crest shown in Fig. 16, determine the weight W per foot of length in order that the crest will drop when the water reaches height B.

29. A gage pressure of 0.5 in. of water is to be measured by an open manometer with the connecting line filled with air at 90 deg. Fahr. Barometric pressure measured at the manometer is 29.9 in. of mercury. The manometer is 100 ft. below the pressure connection. Compute the percentage error in the gage pressure if no correction is made for the effect of the air in the connecting line. What is the atmospheric pressure at the elevation of the pressure connection if the air temperature at the manometer is 68 deg. Fahr.?

CHAPTER II

FLOW PHENOMENA

In the preceding chapter, dealing with fluids in static equilibrium, a number of problems were considered in which movement of the fluid occurred. This naturally raises the question as to what is meant by "flow." In setting up the equations for static equilibrium, the absence of tangential stresses was assumed, and this condition presupposes that the motion occurs without distortion of the fluid. However, this distinction does not exclude the theoretical case of frictionless flow in which no tangential stresses occur in any event. A comprehensive and rigorously correct definition of flow which will exclude problems of static equilibrium is difficult to frame, but from a consideration of the problems in which it evidently occurs, flow may be said to include all cases in which the fluid at a short distance from a solid boundary moves relative to that boundary.

Before taking up the quantitative relationships of subsequent chapters, a few common phenomena will be described in qualitative terms to aid in visualizing the physical problems involved and to provide a background for the necessary assumptions. A quantitative treatment of some of the phenomena described in this chapter is beyond the scope of this text but, nevertheless, these and even more complex situations occur in nature and even a qualitative understanding is important. In this connection it should be remarked that instruction in the applied sciences tends to emphasize problems susceptible to exact analysis and to pass over the great number of common phenomena which have not been solved quantitatively. In engineering practice, problems arise regardless of whether or not they can be solved analytically, and sound judgment based on experience is important because in many cases approximate solutions can be obtained if proper simplifying assumptions are made. As an aid to the analysis of flow problems, a few of the more common phenomena of flow are described in the remainder of this chapter.

11. Velocity at a Solid Boundary.—One difficulty in visualizing the flow of real fluids lies in the movement immediately adjacent to a solid boundary. The theoretical treatment of the basic flow relationships assumes a fluid, devoid of all frictional resistance, which slides along the confining boundaries much as would a rigid body. The transition from ideal to real fluids is then made by means of empirical coefficients without much explanation of the basic physical changes involved. The two extremes of highly rarefied gases and liquids composed of large molecules being neglected, the layer of molecules of a fluid adjacent to a solid boundary is at rest relative to the boundary. The molecules themselves are in motion, but a statistical treatment of the motion resulting from their collisions with the boundary shows that the average conditions are the same as though the fluid were a continuous medium with zero velocity at the boundary.[1]

Recognition of the fact that all real fluids exhibit a condition of zero relative velocity at all solid boundaries has brought about a fundamental change in the methods of analyzing many flow problems and has resulted in a much closer agreement between theory and experiment. At a distance from a solid boundary, the frictionless and actual flow conditions may be very nearly the same, but near the boundary the assumption of frictionless flow results in conditions entirely different from those in a fluid having a small but appreciable viscosity. Since most flow problems are concerned with the effect of solid boundaries of one kind or another, it is to be expected that the difference in viewpoint leads to different analytical results.

12. Turbulence.—A more detailed discussion of turbulent flow is included in a subsequent chapter, but a general description of this phenomenon will shed some light on the nature of the assumptions made in dealing with the theoretical case of frictionless flow. In most of the problems in the motion of air and water, the flow is turbulent in the sense that there occur erratic variations in direction and velocity (Fig. 17). Color injected into a turbulent flow quickly diffuses through it and the motion of light particles is highly irregular. Direct measurements of these fluctuations show that they do not exhibit a regular period and one may well ask what is the significance of the basic energy

[1] E. C. Bingham, "Fluidity and Plasticity," 1st ed., p. 150, McGraw-Hill Book Company, Inc., New York, 1922.

equations, derived on the assumption that the path of the flow can be traced, in a flow exhibiting such random characteristics. A partial answer to this question is that at each point in the flow, the average velocity, direction, and pressure can be measured and one can construct a fictitious flow pattern out of these average vectors. The system of flow tubes considered in the derivation of the Bernoulli energy equation may then be considered as equivalent to this average-flow system and corrections can be made as needed to account for the effect of turbulence. Considering the complex nature of turbulent flow, it is surprising how well the theoretical equations fit experimental data. In the case of the venturi meter, for example, the computed pressure difference between the inlet and the throat agrees very closely with measurement. In other words, the assumption of an idealized situation which does not correspond with reality leads to relationships which are of practical importance but application of these relationships should be made with the understanding that they are close approximations only.

FIG. 17.—Velocity fluctuation in an air stream as measured at California Institute of Technology. The lower line is a time scale.

It should be noted that the energy of turbulence is not available for maintaining the flow. It represents a degraded form of mechanical energy in process of being transformed into thermal energy and so, from the standpoint of a flow problem, may be

considered as lost. In the case of a jet issuing from an orifice, the potential energy must supply the energy of turbulence and, for this reason alone, the energy of the mean flow will be less than the theoretical value. However, in the case of the flow of liquids in long pipes, the energy of turbulence must be nearly the same at all sections and so can be neglected in estimating the pressure drop except as it affects the friction loss.

Another effect of turbulence is in the alteration of the pattern of the average flow. It is known from experiment that the main flow stream passing a sphere or any other curved surface has a tendency to break away from the surface as the velocity is increased. The exact condition under which this separation occurs is sensitive to the degree of turbulence in the stream of fluid approaching the object. This point of separation is related to the dynamic force on the object and in fact the resistance of a sphere to the flow of a fluid past it is used as a quantitative measure of turbulence.

13. Surging.—In addition to the short-period fluctuations in velocity and pressure which characterize turbulent flow, nearly all real flows exhibit longer period variations which will be referred to as surges. Open bodies of water such as lakes, rivers, and canals are commonly observed to oscillate more or less as a unit. Pipelines to and from pumps, blowers, and turbines show these mass movements of varying intensity. Such surging is so commonly observed that steady flow, in which the velocity and pressure show the same average value at all times, is the exception rather than the rule. However, the derivation of the basic equations of flow assumes steady conditions and the question arises as to what must be the magnitude of the surging before its effect is appreciable. Friction losses are approximately proportional to the square of the average velocity in a pipe or canal and, consequently, the loss of energy during surging will exceed that during steady flow at the same temporal mean velocity. One can estimate the magnitude of this difference and in most instances it is not important. In addition to affecting the friction losses, surging adds to the difficulty of obtaining representative measurements of average pressure and velocity. The problem is especially important in measurements of friction losses in rivers where the amplitude of the surging is often of the same order of magnitude as the average drop in the surface elevation.

There are many causes of surging, such as variations in the wind velocity or atmospheric pressure over free water surfaces, dynamic characteristics of pumps or turbines, variations in power supply, vibrations of gates and other control structures, and so forth. An initially steady flow may become periodic and give rise to surging without the application of periodic external forces. A familiar example is the dripping of a water faucet, in which an initially steady flow becomes periodic under the action of surface tension. Siphon spillways may have the same effect if the rate

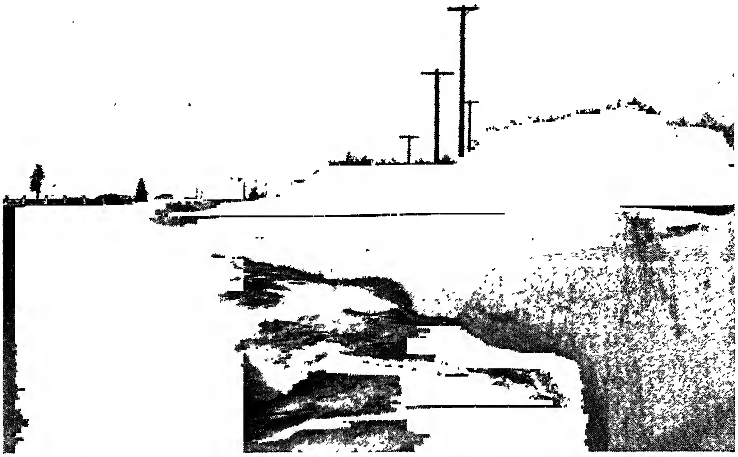

FIG. 18.—Standing waves at a bend in a canal. (*Courtesy of Fred C. Scobey, U. S. Bureau of Agricultural Engineering.*)

of inflow to the upper pool is between the rate at which the siphon primes and the minimum rate of discharge as a siphon. The siphon "makes" and "breaks" intermittently, causing a periodic flow downstream and surging of the upper pool.

14. Waves.—The formation of wind waves on free water surfaces is a commonly observed phenomenon. In general, waves may form at any surface of separation between fluids of different physical properties, especially between fluids of different densities. Such waves advance relative to the fluid with a velocity which depends upon the dynamical conditions and if they are to remain stationary it is necessary that the component of absolute velocity of the fluid be equal and opposite to the relative velocity of the wave. Figure 18 shows standing waves at a bend

in a canal. Similar waves generally form near obstructions which pierce a free water surface. A number of dynamically different types of waves are possible and it is only intended in the present section to point out that waves may form and that this possibility should not be overlooked in the analysis of problems involving a surface of separation between two fluids. This is particularly important in problems of the flow of water in open channels.

15. Curvature of Flow and Pressure Variations.—In the chapter dealing with hydrostatics, the pressure variations were shown to be related to the components of extraneous force and to the accelerations of the fluid mass as a whole. Deviations from these hydrostatic relationships generally result in curvature of flow. For instance, in a free jet of liquid moving through air there is no vertical pressure gradient to sustain the weight of the fluid and a downward acceleration occurs. In curved pipes, curvature of the stream paths is associated with a pressure gradient. The proper assumptions to be made in a number of problems depend upon a knowledge of the relationship between pressure gradient and curvature.

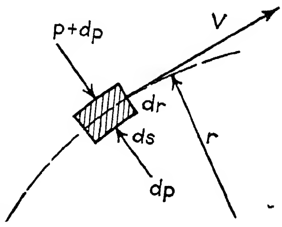

FIG. 19.

Referring to Fig. 19, which represents the path of a fluid particle in the interior of a mass of fluid, the force relationships are, letting dn be the thickness normal to the plane of the figure

$$(p + dp)(dn\ ds) - p\ dn\ ds = \rho\frac{V^2}{r}dn\ ds\ dr$$

Dividing by $dn\ ds\ dr$,

$$\frac{dp}{dr} = \rho\frac{V^2}{r}$$

The pressure gradient is inversely proportional to the radius of curvature of the actual path of the fluid mass. It is important to distinguish between this radius of curvature and that of the boundaries because in many flow problems in which curvature occurs, it is not correct to assume that the fluid follows the

curvature of the boundaries. In fact, in most problems of curved flow, prediction of the actual paths is difficult if not impossible; but nevertheless, a qualitative knowledge of the effect of curvature is useful.

As an example, consider a jet discharging from a sharp-edged orifice (Fig. 20). Three facts regarding the nature of the phenomenon can be obtained from a consideration of the pressure-curvature relationship. First, at the orifice edge, the flow must be tangent to the face because the fluid has been flowing along the face of the orifice plate and, were the initial direction not tangent to the face, r would equal zero and dp/dr would equal ∞. Secondly, since the initial direction is along the face of the plate, convergence of the flow must occur, and therefore the pressure in the interior of the fluid in the plane of the orifice must exceed the pressure p_0 surrounding the jet. Lastly, excess pressure exists in the jet until the stream paths become parallel, and at this point the pressure throughout the jet equals the external pressure, p_0.

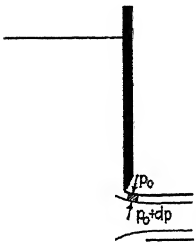

FIG. 20.

For both frictionless and nonfrictionless flow in horizontal paths, the pressure variation in vertical sections follows the hydrostatic relationship. Gravity acts upon the fluid element and the upward thrust of the pressure force must equal the weight if the particle is to continue to move horizontally. Considering unit horizontal area,

$$\begin{aligned} p - (p + dp) &= w\,dz \\ dp &= -w\,dz \end{aligned}$$

The same relationship holds for frictionless flow in inclined pipes but is complicated slightly when the effect of friction is included.

16. Pressure Measurements.—In all types of measurements, a basic problem is the design and use of instruments in such a way that the presence of the measuring instrument does not affect the phenomenon and modify the magnitude of the quantity to be measured. This problem is especially important in the measurement of pressures in fluids. Pressure-measuring devices can be relied upon to indicate the average pressure existing at the point at which the pressure area is in contact with the fluid

stream, but one cannot be certain that the average pressure indicated is that which would exist at the same point had no pressure connection been made. For instance, if, in the installation of a pressure tap in a pipeline, a burr is left along the edge of the opening, the pressure in the opening itself may be too great or too small depending upon the way in which this obstruction changes the local direction of flow. Another difficulty is that the pressure measured may not be that which is applicable to the problem. Pressure connections installed along a concave boundary may give correct readings but the indicated pressures are in excess of the weighted average, which would enter the energy equation for the pipe as a whole. If one is concerned with the forces acting on a boundary, such pressure measurements are applicable, but in most flow problems it is the average pressure in a cross section which is pertinent. For example, in field tests of pumps and blowers, computation of the head developed requires measurement of the average suction and discharge pressures, and care must be exercised to locate the pressure connections so as to give the proper value.

It is to be noted that the pressure differences resulting from both incorrect construction and incorrect location of the pressure opening are functions of the local geometry of the solid boundaries and the local velocity head. The importance of proper pressure connections increases as the velocity head becomes greater in proportion to the differences in pressure head and elevation.

17. Separation and the Formation of Eddies.—Separation has already been mentioned in connection with turbulence. It is a relatively common phenomenon and one which is a frequent cause of differences between theory and experiment. In general terms, it is simply the breaking away of the main fluid stream from the solid boundaries and may result either from the momentum of the stream itself or from the action of an adverse pressure gradient or from both. Taking an extreme example, the discharge flowing from a pipe or canal into a reservoir does not expand so as to fill the reservoir, but rather it expands at a small angle, leaving the solid boundaries entirely. Figure 21 shows separation from a cylinder and from the upper surface of an airfoil. In the expanding section shown in Fig. 22, the pressure increases in the direction of flow because of the decrease in velocity and this adverse pressure gradient soon overcomes the

momentum of the slow-moving particles near the wall. At some point A, the main stream separates from the boundaries and

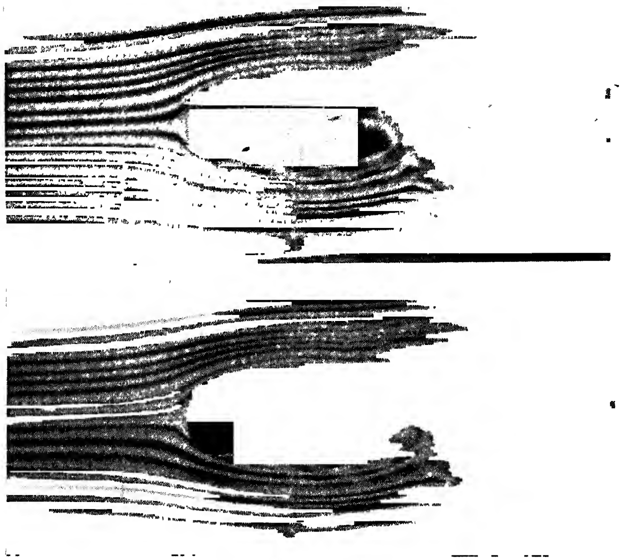

FIG. 21a.

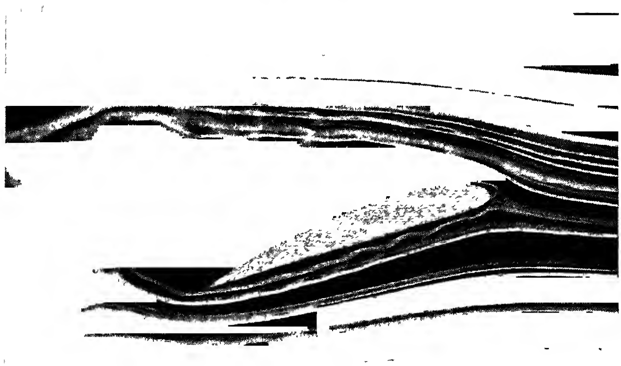

FIG. 21b.

FIG. 21.—Smoke photographs showing separation from objects. Air velocity, 7 ft. per sec. *a*, Separation from a cylinder; *b*, separation from the upper surface of an airfoil. (*Courtesy of National Advisory Committee for Aeronautics.*)

beyond this point a reverse flow occurs under the action of the pressure gradient. Separation is associated with a loss of

energy through the creation of eddies, and analysis of the phenomenon by means of the usual flow equations is difficult because the space occupied by active fluid is unknown.

The eddies associated with separation vary greatly in both size and general characteristics, depending upon the velocities, pressures, and curvatures involved. In rivers and other large bodies of moving water, separation of the main stream from the banks is a common phenomenon. The large eddies formed are relatively constant in position and velocity, and their general form can often be estimated. There is also formed a secondary eddy system at the line of contact between the stream and the large eddy. This small-scale system is irregular in location, size, and velocity.

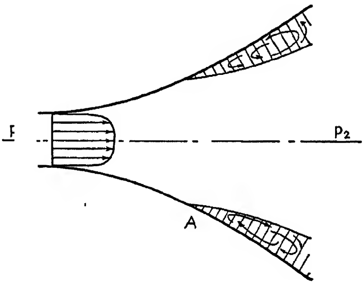

FIG. 22.

An example of the formation of a small-scale eddy system which has been analyzed theoretically is the staggered system of vortices in the wake of a flat plate. Two rows of vortices or small eddies form with a spacing depending on the width of the plate (see Chap. VI). Similar systems are commonly observed behind all kinds of obstructions in both closed and open channels.

An important feature of the formation of eddies is that they involve a loss of energy in addition to the ordinary friction loss of the main stream. Fixed eddies of large dimensions dissipate energy because of their motion relative to the solid boundaries. Small-scale eddies which move with the stream involve the degradation of kinetic energy of directed motion into unavailable energy of rotation. Hence it appears that the creation of both types of eddies results in an increase in the resistance offered to the flow of the main stream, since this is the only available source of energy.

18. Cavitation.—In deriving the equations for the flow of fluids, it is assumed that the fluid is homogeneous, at least in the sense that there is a known relationship between the pressure and volume. One important cause of a departure from the assumed condition of homogeneity is the formation of *cavities*

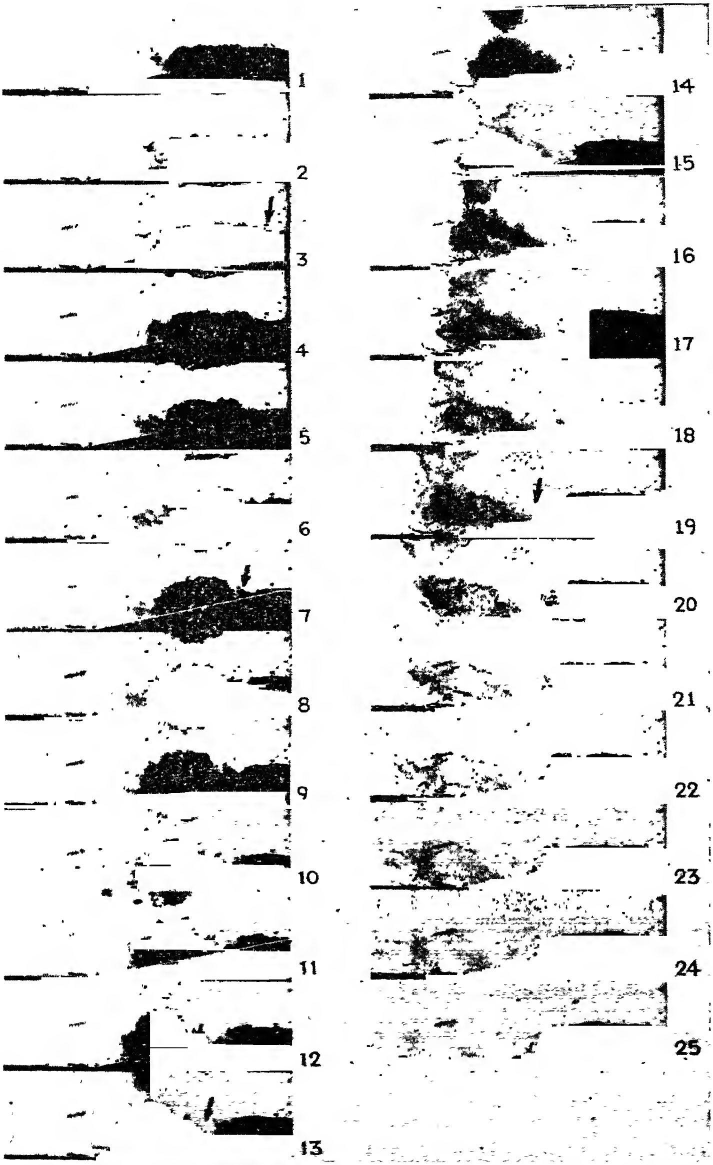

FIG. 23.—Development and decay of cavities in a stream of water. Flow is from right to left. (*Courtesy of J. C. Hunsaker.*)

at low pressure points in liquid systems. Under normal flow conditions, liquids are incapable of remaining continuously at pressures below the vapor pressure of the liquid. If the pressure is reduced to this point, the liquid breaks apart and cavities form which are filled with vapor and dissolved gases. Figure 23 shows the formation and subsequent decay of a cavity caused by the decrease of pressure at a contraction in a pipeline.

In addition to the fact that the formation of cavities renders the usual flow equations invalid, they are undesirable for several other reasons. At the point of cavitation, further reductions in pressure are impossible and this condition frequently fixes the maximum possible rate of discharge. Examples of this phenomenon are the familiar "cutoff" points in the characteristics of centrifugal and jet pumps. Another feature is that the formation of cavities adversely affects the flow pattern and results in a loss of energy. Finally, as the fluid carries the cavities into regions of higher pressure, the vapor abruptly changes back to the liquid phase and the hammering effect involved in the collapse of the cavities is very destructive. In fact, the first studies of the phenomenon of cavitation were made to determine the cause of the severe pitting shown by ships' propellers. These investigations showed that the collapse of a cavity may result in instantaneous pressures as high as 100 atm.

19. Stagnation Points.—In the theoretical treatment of flow around obstructions of any kind, it is found that one of the streamlines shows a continual decrease in velocity and finally comes to rest at the surface at a point known as the *stagnation point.* A similar phenomenon occurs in real flows and can be visualized by neglecting turbulent fluctuation and considering the system of averaged velocities. One such point exists on the surface of every object exposed to a fluid stream which divides and flows around it. An important feature of these stagnation points is that the pressure increase is dependent on the velocity at a distance, and hence this pressure rise may be used as a means of measuring the velocity. The magnitude of the pressure rise appears to be independent of the shape of the obstruction but the location of the stagnation point is often difficult to predict. However, on an object which is symmetrical about an axis parallel with the direction of flow, the stagnation point occurs at the forward end of the axis of symmetry. This fact is utilized in

velocity enough so that the jet strikes too near the foundations. At 90 cu. ft. per sec., however, the jump is "washed out" and the flow will then pass through the bucket without a jump down to 40 cu. ft. per sec.

4. Figure 25 shows "slug" flow in a steep chute under what appeared to be steady conditions in the approach channel. A similar phenomenon is reported to have occurred in some of the steep floodways near Los Angeles, Calif. Surges of small amplitude in the channel above the chute may cause these slugs to

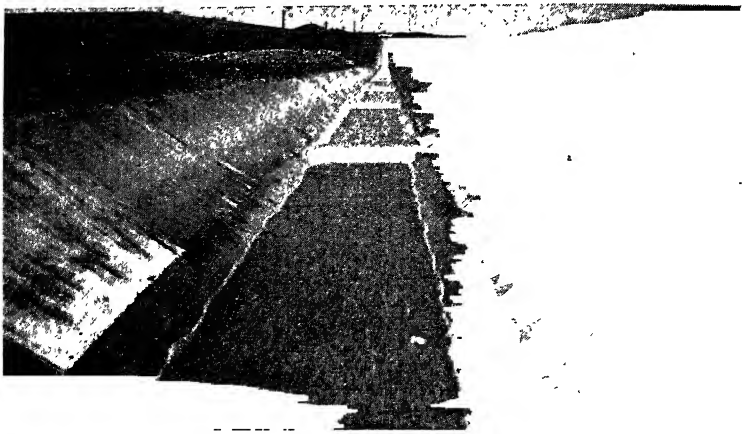

FIG. 25.—"Slug" flow in a steep chute. Similar flow on a much smaller scale may often be seen in street gutters. (*Courtesy of Fred C. Scobey, U. S. Department of Agricultural Engineering.*)

form, but whatever may be the cause, there is an uncertainty of regime involved which should not be overlooked.

5. Another example was observed during some laboratory experiments on the hydraulic jump in trapezoidal channels. It was found that the upstream edge of the surface roller was always inclined at a certain angle with the channel axis but that the leading side could be changed permanently at will by temporarily obstructing the flow on that side.

Other examples might be cited. The point involved is that there may be several possible flow regimes corresponding to a given set of external conditions. The regime which will exist may be determined by imperceptible differences in the flow conditions, such as minute surging or by the sequence of preceding conditions.

CHAPTER III

FRICTIONLESS FLOW

This chapter is devoted to elementary derivations of the equations describing the frictionless motion of a fluid in a unidirectional flow system. The fact that no real flows occur without friction losses detracts but little from the importance of these equations because friction losses are often very small, as in the case of discharge over a weir, or can be estimated accurately. To simplify the equations, the treatment has been restricted to systems in which the flow at any section is predominantly in one direction. For a more precise and detailed derivation applicable to flow in three dimensions, the student should consult one of the standard texts on hydrodynamics.

The basic-flow equations are of such great importance that they are derived in several ways to bring out the significance of the individual terms. Each successive derivation includes the preceding ones as special cases, a procedure involving some repetition but one which appears to be desirable to emphasize the relationship between the "hydraulic" equations and the total-energy equation.

As used in this text, the term *mechanical energy* includes potential and kinetic energy and the term representing the rate at which work is being done on the system by external forces. Thus the Bernoulli equation, as used in hydraulics, deals with mechanical energy only, whereas the total-energy equation, which is necessary in treating the flow of steam, includes thermal energy as well. This total-energy equation is always applicable and its use is recommended to students in solving problems, even though in practical work one chooses at the outset the simplest form consistent with the conditions of the problem to be solved.

22. Equation of Continuity.—Consider a closed tube such as appears in Fig. 26. Application of the principle of the conservation of matter leads to the equation:

$$A_1V_1\rho_1 - A_2V_2\rho_2 = \frac{dM_{1-2}}{dt}$$

Here M represents the mass of fluid between sections (1) and (2), V is the mean velocity, and A is the area of a surface which is at each point perpendicular to the direction of flow. The equation states simply that the mass of fluid entering the region between points (1) and (2) in unit time, less that leaving, must equal the rate at which mass is accumulating in the region. During steady flow the velocity, density, and pressure at each point do not change with time and for this situation

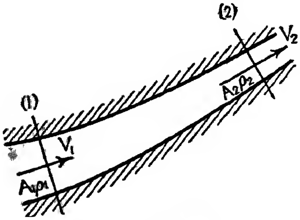

FIG. 26.

$$\frac{dM_{1-2}}{dt} = 0$$

The equation of continuity then becomes

$$A_1V_1\rho_1 = A_2V_2\rho_2 = \cdot\cdot\cdot = A_nV_n\rho_n \tag{3.1}$$

For incompressible fluids,

$$\rho_1 = \rho_2 = \cdot\cdot\cdot = \rho_n$$

and

$$A_1V_1 = A_2V_2 = \cdot\cdot\cdot = A_nV_n$$

A more detailed treatment of the continuity equation will not be attempted because Eq. (3.1) is adequate for the great majority of flow problems. Furthermore, any derivation must proceed from the principle of the conservation of matter, and the proper equations for special conditions are best derived as needed. Branching pipes, chambers in which fluid is accumulating, and other situations lead to equations differing slightly in appearance but basically identical with those stated here.

23. Streamlines and stream tubes are fictions invented for simplification of the theoretical treatment of flow problems. Lines drawn in a fluid from point to point, in such a manner that at any instant their direction is everywhere that of the fluid, are called *streamlines*. In steady flow, the particles follow streamlines. The infinite group of streamlines passing through

any closed curve will form a tube called a *stream tube.* The flow inside a stream tube is supposedly the same as frictionless flow in a rigid tube of the same shape. The difficulty about using the conception of streamlines or stream tubes is that most of the real flows dealt with are turbulent with the fluid particles moving at random between the solid boundaries. Under such circumstances, visualization of a streamline becomes difficult. No error is involved if one merely substitutes "frictionless pipe" wherever the terms *streamline* or *stream tube* are used.

24. Mechanical Energy or Bernoulli Equation.—Considering only mechanical energy, the principle of the conservation of energy may be stated as follows: The work done on a mass of fluid, less the work done by it, is equal to its change in potential

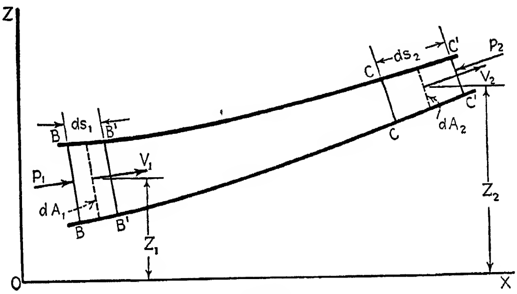

FIG. 27.

and kinetic energy. This statement may be illustrated by reference to Fig. 27. The fluid is considered to be *incompressible,* and in *steady motion.* The lines $BB'CC'$ represent the paths of individual particles, or streamlines, and the areas dA_1 and dA_2 are bounded by the intercepts of such streamlines with a warped surface everywhere perpendicular to the streamlines. In other words, the motions of the particles are perpendicular to the areas chosen, and all of the fluid passing through dA_1 ultimately passes through dA_2.

If we consider the fluid initially filling the stream tube between B and C, at the end of a short interval of time it will have advanced to a new position $B'C'$. Since the flow is steady the conditions between B' and C are the same as at the beginning of the interval and so the net change is equivalent to transferring a mass of fluid from BB' to CC'.

The work done on the fluid during the interval by the pressure p_1 is $p_1\, dA_1\, ds_1$, and the work done by the fluid against the pressure p_2 is $-p_2\, dA_2\, ds_2$. Pressures acting along the solid boundary act at right angles to the direction of motion and do no work. The change in potential energy is

$$z_2 w\, dA_2\, ds_2 - z_1 w\, dA_1\, ds_1$$

where z is the elevation above some arbitrarily assumed datum plane. The change in kinetic energy is

$$w\, dA_2\, ds_2 \frac{V_2^2}{2g} - w\, dA_1\, ds_1 \frac{V_1^2}{2g}$$

Equating the net work done to the total change in energy,

$$p_1\, dA_1\, ds_1 - p_2\, dA_2\, ds_2 = z_2 w\, dA_2\, ds_2 - z_1 w\, dA_1\, ds_1 + w\, dA_2\, ds_2 \frac{V_2^2}{2g} - w\, dA_1\, ds_1 \frac{V_1^2}{2g}$$

It has been stated that the quantity of fluid BB' equaled that of CC' or

$$dA_1\, ds_1 = dA_2\, ds_2$$

By substituting $dA_1\, ds_1$ for $dA_2\, ds_2$ in the above equation and dividing by $w\, dA_1\, ds_1$, the weight of the fluid, there results

$$\frac{p_1}{w} - \frac{p_2}{w} = z_2 - z_1 + \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

Rearranging the terms,

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} \tag{3.2}$$

Since points B and C were chosen at random in the stream tube,

$$\frac{p}{w} + z + \frac{V^2}{2g} = \text{const.} \tag{3.3}$$

This is known as Bernoulli's equation and is in the form in which it is ordinarily used in hydraulic computations.

It is interesting to note that if the velocity does not vary along the stream tube, the last equation can be written:

$$\frac{p}{w} + z = \text{const.}$$

Differentiating,

$$dp = -w\,dz$$

which will be recognized as equivalent to the equation of pressure variation in a static liquid [Eq. (1.6)].

Bernoulli's equation may also be derived by a consideration of the forces acting on an elementary stream tube having a cross-section area dA and a length ds (Fig. 28).

In any stream tube, the velocity may vary continuously from point to point along the tube and may also vary with time at a particular point. Considering an individual particle, the change in velocity which it experiences may be broken up into two parts: namely, the rate of change of velocity along a stream tube *at*

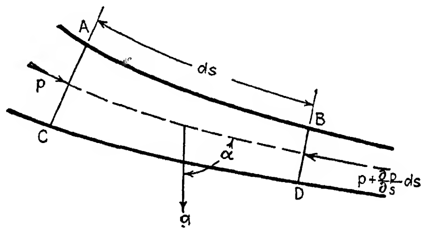

FIG. 28.

any instant and the rate of change with time *at a point*. Expressing this by differential coefficients gives,

$$dV = \left(\frac{\partial V}{\partial s}\right)_t ds + \left(\frac{\partial V}{\partial t}\right)_s dt$$

where dV = the change in velocity of a particle with respect to both position and time,

$(\partial V/\partial s)_t$ = the rate of change in velocity along a streamline at any particular time,

$(\partial V/\partial t)_s$ = the rate of change in velocity with respect to time at any point.

The forces acting on the elementary mass are the force of gravity and the pressures. Since we are dealing with a perfect fluid, no tangential stresses can be transmitted by the walls of the stream tube. The force due to gravity is $-g \cos \alpha\rho\, ds\, dA$. The force in the direction of motion due to pressure is $p\,dA$ and in the opposite direction is

$$-\left(p + \frac{\partial p}{\partial s}ds\right)dA$$

The net force is thus

$$-\frac{\partial p}{\partial s}ds\ dA\ -\rho g\cos\alpha\ ds\ dA.$$

The inertial reaction of the mass of fluid is the mass times the acceleration. The acceleration may be written

$$\frac{dV}{dt}=\frac{\partial V}{\partial s}\frac{ds}{dt}+\frac{\partial V}{\partial t}=V\left(\frac{\partial V}{\partial s}\right)+\frac{\partial V}{\partial t}=\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)+\frac{\partial V}{\partial t}$$

and the resisting inertial force is

$$\rho\ ds\ dA\left[\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)+\frac{\partial V}{\partial t}\right]$$

Equating this expression to the net force in the direction of motion,

$$-\frac{\partial p}{\partial s}ds\ dA\ -\ \rho g\cos\alpha\ ds\ dA\ =\ \rho\ ds\ dA\left[\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)+\frac{\partial V}{\partial t}\right]$$

Dividing by the mass, $\rho\ ds\ dA$,

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}-g\cos\alpha=\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)+\frac{\partial V}{\partial t}$$

But $\cos\alpha=\partial z/\partial s$ and the equation can be written

$$\frac{1}{\rho}\frac{\partial p}{\partial s}+g\frac{\partial z}{\partial s}+\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)+\frac{\partial V}{\partial t}=0 \tag{3.4}$$

Equation (3.4) is the general equation of motion for unidimensional flow. It may be simplified somewhat by assuming that the flow is steady, *i.e.*, that it does not vary with time. Then $\partial V/\partial t=0$ and the equation becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial s}+g\frac{\partial z}{\partial s}+\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)=0$$

Integration along the streamline in the direction of s yields

$$\int\frac{\partial p}{\rho}+gz+\frac{V^2}{2}=\text{const.} \tag{3.5}$$

which is Bernoulli's equation modified so as to apply to compressible fluids. The integral may be evaluated if it is possible to

state the relationship between p and ρ. A further simplification may be made by assuming the fluid incompressible (ρ = const.). Then Eq. (3.5) becomes, after dividing by g,

$$\frac{p}{w} + z + \frac{V^2}{2g} = \text{const.} \tag{3.3}$$

which has previously been obtained by considerations of energy.

It will be well at this point to examine the dimensions of the terms in Eq. (3.3).

$$\frac{p}{w} = \frac{\frac{F}{L^2}}{\frac{F}{L^3}} = L$$

$$z = L$$

$$\frac{V^2}{2g} = \frac{\frac{L^2}{T^2}}{\frac{L}{T^2}} = L$$

Each term has the net dimension of a length and is commonly referred to as *head; i.e.*, pressure head, potential or elevation head, and velocity head. From the steps in the derivation it is apparent that z and $V^2/2g$ represent energy per unit weight while p/w represents the rate at which work is being done on the system per unit weight of fluid. Equation (3.3) applies only to steady flow and for this condition; the term p/w may be used as though it also represented an energy possessed by each unit weight.

The significance of the term p/w really deserves more attention than it has received. To illustrate the type of difficulty which may arise, consider a large tank from which issues a small but steady flow. Taking the center of the orifice as a reference datum, and assuming the velocity in the tank to be negligible, the energy relationship is

$$\frac{p}{w} + z = \text{const.}$$

This is the kinetic energy of each unit weight of the issuing jet and the question arises as to whether it is the potential energy of each unit weight of fluid within the tank. If it is, then the total energy of the fluid within the tank is the weight of fluid

times $\left(\frac{p}{w} + z\right)$. However, if it is really an energy which may be assigned to each unit of mass, then it should be possible to stop the inflow to the tank and convert all of it into kinetic energy. This is, of course, absurd, because once the inflow stops, the total energy available as kinetic energy in the issuing jet is measured by the vertical distance from the center of gravity of the liquid to the center of the orifice. Errors of this kind may be avoided by regarding the term p/w as representing the work of external forces per unit weight rather than as an energy term of the same nature as z or $V^2/2g$.

Another misconception regarding the term p/w is that it represents strain energy or energy of compression. There is no step in the derivation which even suggests this idea and it is mentioned only to warn the student against it.

To many the preceding discussion of the term p/w may appear to be unnecessary and perhaps it is. However, there are textbooks on hydraulics which "derive" Bernoulli's equation by simply stating that there are three kinds of head—pressure, potential, and velocity—and that the principle of the conservation of energy shows that the sum of these heads is a constant. Derived in such a manner, the equation appears to be unrelated to ordinary mechanics, and furthermore, if the statements regarding it are applied literally, serious errors may result.

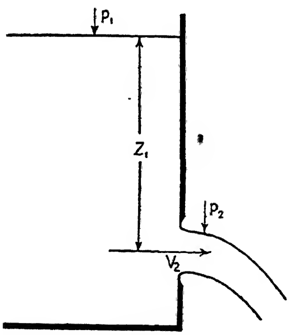

FIG. 29.

25. Application to Liquids. 1. *Torricelli's Theorem.*—The equation for the steady, frictionless discharge from an orifice is one of the first known quantitative relationships in hydraulics. Referring to Fig. 29, the velocity through the tank is so small that the velocity head is negligible. Since a liquid is involved, atmospheric pressure need not be considered. Taking the datum of elevation at the center of the orifice, Bernoulli's equation [Eq. (3.2)] becomes

$$\frac{p_1}{w} + z_1 + 0 = \frac{p_2}{w} + 0 + \frac{V_2^2}{2g}$$

Then, since $p_1 = p_2$,

$$V_2 = \sqrt{2gz_1}$$

which is the velocity of efflux of any liquid. It is also applicable to the discharge of gases if z_1 represents the pressure difference in feet of the gas, and if this difference is small in comparison with the absolute pressure.

As an example, consider a tank of water, with its surface under atmospheric pressure, which discharges into a closed chamber at a point 15 ft. below the free surface. The pressure in the closed tank is 5 lb. per sq. in. gage. The basic equation becomes

$$0 + 15 + 0 = \frac{5 \times 144}{62.4} + 0 + \frac{V_2^2}{2g}$$

$$V_2 = 14.92 \text{ ft. per sec.}$$

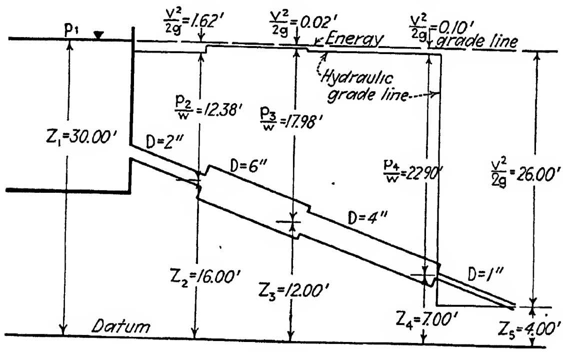

FIG. 30.

If the liquid were oil with a specific gravity of 0.9, the equation would be

$$0 + 15 + 0 = \frac{5 \times 144}{0.9 \times 62.4} + 0 + \frac{V_2^2}{2g}$$

$$V_2 = 11.84 \text{ ft. per sec.}$$

The difference between water and oil in this case results from the fact that the same back pressure represents a greater head of oil than of water. If the heads in feet of the fluid flowing were the same, the theoretical velocities would have been identical regardless of the specific gravity.

The student should note particularly that head is expressed in terms of *the fluid flowing.*

2. *Steady Flow through a Series of Pipes.*—Figure 30 represents a pipe system which is discharging water from a large reservoir at (1) into the air at (5). We may write Bernoulli's equation for each point in the pipe:

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} = \frac{p_3}{w} + z_3 + \frac{V_3^2}{2g} = \frac{p_4}{w} + z_4 + \frac{V_4^2}{2g} = \frac{p_5}{w} + z_5 + \frac{V_5^2}{2g}.$$

By continuity,

$$A_1V_1 = A_2V_2 = A_3V_3 = A_4V_4 = A_5V_5$$

and

$$V_2 = \left(\frac{1}{2}\right)^2 V_5, \qquad V_3 = \left(\frac{1}{6}\right)^2 V_5, \qquad V_4 = \left(\frac{1}{4}\right)^2 V_5,$$

$$p_1 = p_5 = 0$$

Substituting numerical values in the equation,

$$0 + 30 + 0 = \frac{p_2}{w} + 16 + \left(\frac{1}{2}\right)^4 \frac{V_5^2}{2g} = \frac{p_3}{w} + 12 + \left(\frac{1}{6}\right)^4 \frac{V_5^2}{2g} = \frac{p_4}{w} + 7 + \left(\frac{1}{4}\right)^4 \frac{V_5^2}{2g} = 0 + 4 + \frac{V_5^2}{2g}$$

From the first and last portions of this equation,

$$0 + 30 + 0 = 0 + 4 + \frac{V_5^2}{2g}$$

$$V_5 = \sqrt{2g(30 - 4)} = 40.90 \text{ ft. per sec.}$$

The other velocities are then

$$V_2 = \left(\frac{1}{2}\right)^2 \times 40.90 = 10.22 \text{ ft. per sec.}$$

$$V_3 = \left(\frac{1}{6}\right)^2 \times 40.90 = 1.14 \text{ ft. per sec.}$$

$$V_4 = \left(\frac{1}{4}\right)^2 \times 40.90 = 2.56 \text{ ft. per sec.}$$

and the corresponding kinetic energies are

$$\frac{V_2^2}{2g} = 1.62 \text{ ft.}, \qquad \frac{V_3^2}{2g} = 0.02 \text{ ft.}$$

$$\frac{V_4^2}{2g} = 0.10 \text{ ft.}, \qquad \frac{V_5^2}{2g} = 26.00 \text{ ft.}$$

The pressures can now be calculated from the remaining equations:

$$\frac{p_2}{w} = 30 - 16 - \left(\frac{1}{2}\right)^4 \frac{V_5^2}{2g} = 12.38 \text{ ft.}$$

$$\frac{p_3}{w} = 30 - 12 - \left(\frac{1}{6}\right)^4 \frac{V_5^2}{2g} = 17.98 \text{ ft.}$$

$$\frac{p_4}{w} = 30 - 7 - \left(\frac{1}{4}\right)^4 \frac{V_5^2}{2g} = 22.90 \text{ ft.}$$

These pressures when plotted above the pipeline as illustrated in Fig. 30 define the "hydraulic grade line" or the height to which the fluid in the pipe would rise if a manometer connection were made to the pipe at that point. The hydraulic grade line should be distinguished from the "energy grade line" which is above the hydraulic gradient by the amount of the kinetic energy per unit weight, $V^2/2g$, and which is horizontal in the case of frictionless flow in straight pipes, since there is no energy loss. Even in frictionless flow however, losses may occur by "shock" (see Sec. 32).

3. *Venturi Meter.*—The venturi meter is a device used to measure the rate of flow of fluid in a pipeline. It is based on the principle involved in Bernoulli's equation; namely, that an increase in the velocity and kinetic energy of a fluid stream is accompanied by a drop in pressure.

The venturi meter consists simply of two conical sections of pipe joined at their small ends by a short cylindrical section called the *throat*. In some types, particularly those which are small enough to be cast in one piece, the throat is not cylindrical but is a curved section joining the two cones.

Referring to Fig. 31, let the point (1) be a point in the pipeline ahead of the meter and (2) a point in the throat or most constricted section. Assuming frictionless flow, Eq. (3.2) gives

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g}$$

From continuity,

$$V_2 = \frac{A_1 V_1}{A_2}, \qquad \frac{V_2^2}{2g} = \frac{A_1^2 V_1^2}{A_2^2 2g}$$

Substituting and solving for V_1,

$$V_1 = \frac{A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g}\sqrt{\frac{p_1 - p_2}{w} + (z_1 - z_2)}$$

The rate of flow is then

$$Q = A_1V_1 = \frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g}\sqrt{\frac{p_1 - p_2}{w} + (z_1 - z_2)}$$

By letting $h = \frac{p_1 - p_2}{w} + (z_1 - z_2)$ (shown graphically in Fig. 31), the equation can be written:

$$Q = K\sqrt{2gh} \tag{3.6}$$

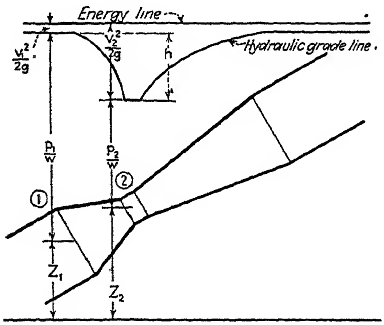

Fig. 31.

where $K = A_1A_2/(A_1^2 - A_2^2)^{\frac{1}{2}}$, a function of the dimensions of the meter and hence a constant. For practical use, since real fluids are not frictionless, K of Eq. (3.6) must include a coefficient which in general depends on the properties of the fluid, the rate of discharge, and the dimensions of the meter. The matter of venturi meter coefficients will be taken up in more detail in the next chapter.

These examples serve to illustrate the application of Bernoulli's equation [Eq. (3.2)] to incompressible flow. Some examples will now be given of its application to the flow of a compressible fluid.

26. Application to Gases. 1. *Venturi Meter.*—The flow is assumed to be steady and Eq. (3.5) is applicable. The converging section is relatively short and the flow may be assumed

to take place adiabatically, with the corresponding relationship between pressure and density given by the equation

$$\rho = \rho_1\left(\frac{p}{p_1}\right)^{\frac{1}{n}}$$

The first term of Eq. (3.5) then becomes

$$\int\frac{dp}{\rho} = \frac{n}{n-1}\frac{p_1}{\rho_1}\left[\left(\frac{p}{p_1}\right)^{\frac{n-1}{n}} - 1\right]$$

and the complete equation between any two points is

$$\frac{V_2^2 - V_1^2}{2g} = \frac{n}{n-1}\frac{p_1}{w_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] + z_1 - z_2 \qquad (3.7)$$

From continuity,

$$A_1V_1\rho_1 = A_2V_2\rho_2$$

and

$$V_2^2 = V_1^2\frac{A_1^2}{A_2}\left(\frac{p_1}{p_2}\right)^{\frac{2}{n}}$$

Furthermore,

$$\frac{p_1}{\rho_1} = gRT_1$$

Substituting these quantities and solving,

$$V_1 = \sqrt{\frac{2\left\{\frac{n}{n-1}\cdot gRT_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] + g(z_1 - z_2)\right\}}{\left(\frac{A_1}{A_2}\right)^2\left(\frac{p_1}{p_2}\right)^{\frac{2}{n}} - 1}}$$

The rate of flow in pounds per second is $W = A_1V_1w_1$ or

$$W = \frac{A_1p_1}{RT_1}\sqrt{\frac{2\left\{\frac{n}{n-1}\cdot gRT_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] + g(z_1 - z_2)\right\}}{\left(\frac{A_1}{A_2}\right)^2\left(\frac{p_1}{p_2}\right)^{\frac{2}{n}} - 1}} \qquad (3.8)$$

This is the theoretical equation for the adiabatic flow of a compressible fluid through a venturi meter. To be applicable practically, it must include a coefficient which is a function of the properties of the fluid, the dimensions of the meter, the rate of discharge, and the ratio of the upstream and throat pressures. Equation (3.8) may be simplified somewhat by assuming $z_1 = z_2$.

It is evident that an equivalent expression should be obtained if Eq. (3.7) had been solved to obtain the throat velocity V_2. The weight rate of discharge would then be $A_2V_2w_2$. The difference in the equations is illustrated by the next section.

2. *Orifices and Nozzles.*—It is evident that the equations for the venturi meter apply to frictionless flow between any two sections of different area. However, to illustrate some of the different forms which they may take, the discharge from a large chamber through an orifice or nozzle will be treated. Returning to Eq. (3.7), suppose that flow occurs from a large vessel in which V_1 is so small that it may be neglected in comparison with V_2. Solving for V_2,

$$V_2 = \sqrt{2g\frac{n}{n-1}RT_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right] + 2g(z_1 - z_2)}$$

If we make $z_1 = z_2$,

$$V_2 = \sqrt{2g\frac{n}{n-1}RT_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right]} \tag{3.9}$$

It is interesting to note that the discharge velocity is a function of the ratio of p_2 to p_1, *i.e.*, the velocity is the same when the fluid expands from a pressure of 5 to 1 atm., as when it expands from 1 to $\frac{1}{5}$ atm.

The weight rate of flow is $W = A_2V_2w_2$. For adiabatic flow,

$$w_2 = \frac{1}{v_1}\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}} = \frac{p_1}{RT_1}\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}}$$

Making the indicated substitutions,

$$W = A_2p_1\sqrt{\frac{2g}{RT_1}\frac{n}{n-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{n}} - \left(\frac{p_2}{p_1}\right)^{\frac{n+1}{n}}\right]} \tag{3.10}$$

It would appear that this result could not be generally applicable since, according to it, the discharge into a perfect vacuum is zero. The discharge must evidently be zero when $p_2 = p_1$ and there must be some value of the ratio p_2/p_1 which gives a maximum rate of discharge. Since all the terms outside the bracket may be considered as constants, the discharge is a maximum when the expression in brackets has a maximum value. Letting $p_2/p_1 = x$ and differentiating,

$$2x^{\frac{2-n}{n}} - (n + 1)x^{\frac{1}{n}} = 0$$

$$x = \frac{p_2}{p_1} = \left(\frac{2}{n + 1}\right)^{\frac{n}{n-1}}$$

For a value of $n = 1.405$, the ratio of p_2/p_1 which gives a maximum discharge is 0.528. At point (2), the velocity V_2 then equals the velocity of sound at the temperature T_2.

Actually, the discharge is not less than the maximum value when p_2/p_1 is less than this critical value. At the maximum discharge, the velocity of the fluid is exactly that of the propagation of a pressure wave and a reduction of pressure below the nozzle is not transmitted upstream through the flowing fluid. The effective pressure ratio therefore remains at that producing maximum discharge, no matter how much the downstream pressure may be reduced.

Application of these equations to flow through a sharp-edged orifice introduces a new type of correction factor, or coefficient, which depends for its value upon inertia effects as well as friction. The issuing stream cannot bend abruptly through 90 deg. at the edge of the orifice and the actual jet area A_2 over which the uniform pressure p_2 occurs is smaller than the area of the orifice itself. The matter will be treated later in connection with orifice coefficients and is mentioned here only to emphasize the fact that the low coefficients for orifices are not caused by friction alone, and that the assumption of frictionless flow is a close approximation to the true flow conditions in orifices as well as venturi meters and nozzles.

27. Total-energy Equation.—Up to this point, the discussion has been concerned solely with mechanical energy. However, a fluid may contain energy in a number of other forms, such as heat energy, electrical energy, chemical energy, or some of the

other forms of atomic or molecular energy. For some flow problems, it is necessary to consider several forms of energy and the conversion from one to another. We shall confine our attention here to two forms, mechanical and heat energy, and their derivative relationships.

The principle of the conservation of energy is a formulation of scientific experience to the effect that energy may exist in many variable and interchangeable forms but may not be quantitatively created or destroyed. The total amount of energy entering any region must equal the amount which leaves the region plus any accumulation, or minus any diminution, of the energy stored within the region.

The relation between heat energy and mechanical energy, in English units, has been found experimentally to be that one British thermal unit (B.t.u.), the amount of heat required to raise the temperature of 1 lb. of water 1 deg. Fahr., is equivalent to 778 ft.-lb. of mechanical energy. Letting J = the mechanical equivalent of heat (778 ft.-lb. per B.t.u.), one form of the principle of the conservation of energy is

$$dq = du + \frac{1}{J}p\,dv$$

or

$$J\,dq = J\,du + p\,dv$$

where q = the heat added to the substance, B.t.u. per pound.
u = the internal energy, B.t.u. per pound.
p = pressure, pounds per square foot.
v = specific volume, cubic feet per pound.
J = mechanical equivalent of heat (778 ft.-lb. per B.t.u.).

Integrating the equation between two points in a flow system,

$${}_1q_2 = u_2 - u_1 + \frac{1}{J}\int_1^2 p\,dv$$

or

$$J(u_2 - u_1) + \int_1^2 p\,dv - J\,{}_1q_2 = 0 \tag{3.11}$$

For gases, the quantity u may be computed from

$$Ju = \frac{1}{n-1}pv \tag{3.12}$$

With respect to mechanical energy, it has already been shown that in steady flow

$$\int_1^2 \frac{dp}{w} + (z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g} = 0 \qquad (3.5)$$

Adding Eqs. (3.5) and (3.11) and remembering that $1/w = v$,

$$\int_1^2 v\, dp + \int_1^2 p\, dv + (z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g} + J(u_2 - u_1) - J_1q_2 = 0$$

The term J_1q_2 represents the mechanical equivalent of the heat added to or taken from the system between points (1) and (2). If ${}_1W_2$ represents the mechanical energy added to or taken from the system between points (1) and (2), then

$$J_1q_2 + {}_1W_2 = {}_1R_2$$

where ${}_1R_2$ denotes the sum of heat or mechanical energy added to or taken from the system. The equation becomes

$$\int_1^2 d(pv) + (z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g} + J(u_2 - u_1) - {}_1R_2 = 0 \qquad (3.13)$$

This is the total-energy equation and includes all the terms representing heat and mechanical energy. The last term, ${}_1R_2$, may be mechanical energy added to a system by a pump or taken from it by a turbine. It may be heat energy added by a boiler or taken away by a condenser. It may be the energy of combustion of a gas. It cannot represent the so-called *loss of energy* due to friction, for in steady flow the walls of a stream tube or pipe are rigid and can transmit no mechanical energy outside the system. The loss of energy through rigid walls can occur only by the transmission of heat. If the walls of a pipe, for example, are considered as perfectly insulated, there is no loss of energy in friction. The effect of friction in such a case is the transformation of mechanical energy into heat. In the flow of a liquid, such as water, this heat energy can seldom be recovered economically and hence is spoken of as "lost." In the flow of gases the situation is different, as heat is more readily converted to mechanical energy.

Whenever the relation between p and v is known, the term $\int v\, dp$ can be evaluated and the conditions of flow can be obtained

from the equation for mechanical energy. In the case of vapors the experimental data are given in the form of tables or diagrams which give the heat content or enthalpy:

$$h = u + \frac{1}{J}pv,$$

or

$$J(h_2 - h_1) = J(u_2 - u_1) + (p_2v_2 - p_1v_1)$$

Substituting in Eq. (3.13), there results

$$(z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g} + J(h_2 - h_1) - {}_1R_2 = 0 \quad (3.14)$$

For gases it is convenient to modify Eq. (3.14) by use of Eq. (3.12) to

$$(z_2 - z_1) + \frac{V_2^2 - V_1^2}{2g} + \frac{n}{n-1}(p_2v_2 - p_1v_1) - {}_1R_2 = 0$$

28. Adiabatic Flow of Steam through a Nozzle.—Applying Eq. (3.14) to the adiabatic flow of steam from a large chamber through a nozzle, the special conditions are

$$z_1 = z_2 = 0, \qquad {}_1R_2 = 0, \qquad V_1 = 0$$

Solving for V_2,

$$V_2 = \sqrt{2gJ(h_1 - h_2)}, \qquad V_1 = 223.7\sqrt{h_1 - h_2}$$

Example: Dry steam under a pressure of 25 lb. per sq. in. abs. flows through a 1-in. nozzle into the atmosphere (14.7 lb. per sq. in.). Using a Mollier chart the following values are obtained:

Point (1):

$$\begin{aligned} \text{Heat of the liquid} &= 208.3 \text{ B.t.u.} \\ \text{Heat of evaporation} &= 951.9 \text{ B.t.u.} \\ h_1 &= 1160.2 \text{ B.t.u.} \\ \text{Entropy} &= 1.7137 \end{aligned}$$

Point (2): From Mollier chart at constant entropy,

$$\begin{aligned} h_2 &= 1121.2 \text{ B.t.u.} \\ V_2 &= 223.7\sqrt{1160.2 - 1121.2} \\ &= 1397 \text{ ft. per sec.} \end{aligned}$$

The amount of water in the steam at the tip of the nozzle is 3 per cent. The specific volume of water at 14.7 lb. per sq. in. is 0.0167 cu. ft. per lb. The

specific volume of vapor at 14.7 lb. per sq. in. is 26.8 cu. ft. per lb. The specific volume of the mixture is

$$v = 26.8 \times 0.97 + 0.0167 \times 0.03 = 26.0 \text{ cu. ft. per lb.}$$

Then the weight rate of discharge is

$$W = \frac{1397 \times \pi}{144 \times 4 \times 26.0} = 0.293 \text{ lb. per sec.}$$

29. Adiabatic Flow of Gas through a Nozzle.—The adiabatic flow of a gas through a nozzle may be obtained from Eq. (3.13) by applying as special conditions,

$$z_1 = z_2 = 0, \qquad {}_1R_2 = 0$$

and since no energy of any sort is added to the fluid during the process,

$$J(u_2 - u_1) + \int_1^2 p\,dv = 0$$

Equation (3.13) becomes

$$\frac{V_2^2 - V_1^2}{2g} = \int^2 v\,dp$$

In an adiabatic process, the relation between p and v is fixed as

$$v = \left(\frac{p_1}{p}\right)^{\frac{1}{n}} v_1$$

and

$$\frac{V_2^2 - V_1^2}{2g} = p_1^{\frac{1}{n}} v_1 \int_2^1 \frac{dp}{p^{\frac{1}{n}}}$$

from which point the analysis is the same as that of Sec. 26.

The equations used for steam and for gases are different in appearance only because of the different manner in which data on the pressure-volume relationship are presented. Because of the importance of steam and the peculiarities of its physical properties, values of h are presented in graphical or tabular form and it is convenient to put the flow equations in such form as to use these values directly.

30. Momentum Equation.—In fluid mechanics, as in rigid-body mechanics, it is frequently necessary to solve problems in

which all the details of the process involved are not entirely clear. In such cases, it may be difficult, if not impossible, to apply the energy equation. In a sudden expansion in a pipeline, for example, there is a loss of mechanical energy and an increase of pressure. The rise of pressure cannot be calculated from the energy equation because the amount of the loss is unknown. In this case and in many others, recourse may be had to the principle of momentum in which only the conditions along the boundaries of a volume need be considered.

Momentum is defined as the product of a mass and its velocity. It is related to force by the equation

$$F_s\,dt = d(mV_s) = m\,dV_s + V_s\,dm \tag{3.15}$$

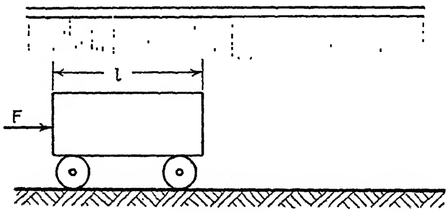

FIG. 32.

in which F_s is the component of the force acting in the S-direction on the mass m, necessary to cause a change of velocity dV in time interval dt, and dm is the change in the mass during the same interval. This equation may be integrated to

$$\int_0^t F_s\,dt = mV_s - m_0V_{s0} \tag{3.16}$$

where m and V_s are the mass and velocity at time t, and m_0 and V_{s0} are the mass and velocity at time 0, respectively.

As an example of the application of this principle, consider the movable tank of Fig. 32. A force F moves the car under a perforated pipe which discharges into the tank. Let the mass and velocity of the tank at time $t = 0$ be m_0 and V_0, and the mass rate of discharge from the pipe per foot of length be q per second. If the length of the tank is l ft., the rate of increase of mass is ql per second, and in time t the total mass is

$$m = m_0 + qlt$$

Equation (3.16) becomes

$$Ft = (m_0 \quad qlt)V -$$

from which the final velocity V can easily be calculated. The process of finding V from considerations of force and acceleration is somewhat more involved.

For the case of steady fluid motion in two dimensions, Eq. (3.15) may be transformed so as to include the rate of flow instead of the mass m and the time dt. Referring to Fig. 33 and considering the section between (1) and (2), the mass entering at (1) equals that leaving at (2) and hence $dm = 0$. If we consider

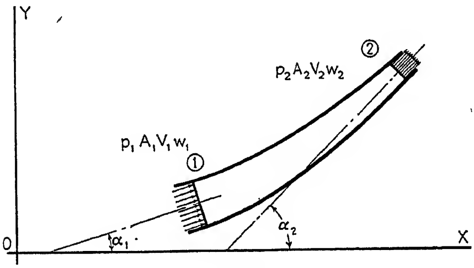

FIG. 33.

only the X-component of the forces acting, Eq. (3.15) may be written

$$F_x \Delta t = m \Delta V_x$$

The mass m undergoing the change in velocity ΔV_x in time Δt is

$$m = \frac{A_1 V_1 w_1}{g} \Delta t = \frac{A_2 V_2 w_2}{g} \Delta t$$

and

$$\Delta V_x = V_2 \cos \alpha_2 - V_1 \cos \alpha_1$$

Making the substitutions indicated,

$$F_x = \frac{A_1 V_1 w_1}{g} (V_2 \cos \alpha_2 - V_1 \cos \alpha_1)$$

For liquids, w is constant, and $A_1 V_1 = Q$ so that

$$F_x = \frac{Qw}{g} (V_2 \cos \alpha_2 - V_1 \cos \alpha_1) \qquad (3.17)$$

Similarly,

$$F_y = \frac{Qw}{g}(V_2 \sin \alpha_2 - V_1 \sin \alpha_1). \tag{3.18}$$

Before developing a more general treatment of momentum relationships, a number of problems will be worked out to stress the basic ideas.

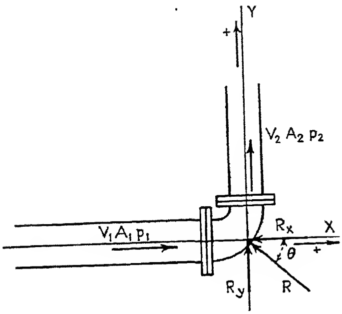

FIG. 34.

31. Force Exerted on Pipe Bends.—Assume that the elbow of Fig. 34 lies in a horizontal plane, that a fluid flows through it in the direction indicated, that the density of the fluid is constant, and that there is no friction. Summing the forces acting in the X-direction [Eq. (3.17)] and assuming that forces acting to the right are positive,

$$F_x = p_1A_1 + R_x = Q\frac{w}{g}(V_2 \cos \alpha_2 - V_1 \cos \alpha_1)$$

In this case $\alpha_1 = 0$, $\alpha_2 = 90°$, $\cos \alpha_1 = 1$, $\cos \alpha_2 = 0$. Making these substitutions,

$$R_x = -p_1A_1 - \frac{Qw}{g}V_1 \tag{3.19}$$

The minus signs indicate that R_x acts toward the left. Similarly for the Y-direction, assuming forces positive when acting upward,

$$F_y = R_y - p_2A_2 = \frac{Qw}{g}(V_2 \sin \alpha_2 - V_1 \sin \alpha_1)$$

Here $\sin \alpha_1 = 0$ and $\sin \alpha_2 = 1$, hence

$$R_y = p_2A_2 + \frac{Qw}{g}V_2 \tag{3.20}$$

The forces R_x and R_y are those exerted on the fluid by the elbow. Actually the velocity and pressure across the section (2) are not uniformly distributed and the correct form of the equation is

$$R_y = \int_0^{A_2} p\, dA + \frac{w}{g}\int_0^{A_2} V^2\, dA$$

However, Eq. (3.20) is a good approximation, especially when the static pressure is large in comparison with the dynamic pressure.

The resultant force acting on the elbow is

$$R = \sqrt{R_x^2 + R_y^2}$$

If the angle made by the resultant with the X-axis is θ (see Fig. 34), then

$$\tan \theta = \frac{R_y}{R_x}$$

32. Sudden Expansion.—A sudden expansion in a pipeline is accompanied by a decrease in the velocity of the fluid flowing and an increase in the pressure. If the velocity and pressure before the expansion are known (p_1 and V_1 of Fig. 35), the velocity after expansion, V_2, may be determined by the continuity relation, if the fluid is incompressible. If the fluid is compressible, the continuity relation includes the density which in turn depends on the pressure. In either case, p_2 cannot be determined by application of Bernoulli's equation since there is an unknown loss of energy in the expansion. The momentum principle may be used to advantage.

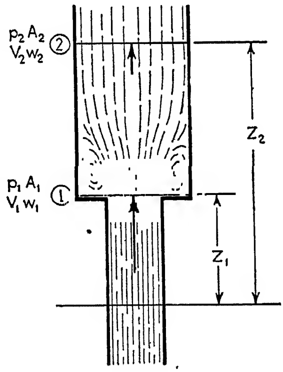

FIG. 35.

Figure 35 represents a small pipe of area A_1 discharging a fluid into a larger pipe of area A_2. Pressures and velocities are as indicated. The flow from the smaller pipe is sensibly parallel for a short distance after leaving the pipe, and hence the pressure on the end of the larger pipe may be assumed to be essentially p_1. If it is assumed that the pipes are vertical, and that there is no friction, the net force acting upward is

$$F_x = p_1A_2 - p_2A_2 - A_2\int_{z_1}^{z_2} w \, dz$$

By Eq. (3.17),

$$F_x = \frac{Q_2w_2}{g}V_2 - \frac{Q_1w_1}{g}V_1$$

Since $W = Q_2 w_2 = Q_1 w_1$,

$$-A_2 \int_{z_1}^{z_2} w\, dz + (p_1 - p_2)A_2 = \frac{W}{g}(V_2 - V_1)$$

The difference between the pressures depends on the relation between p and v. For an incompressible fluid,

$$w_1 = w_2 = w, \qquad A_1V_1 = A_2V_2 = Q$$

and

$$A_2\left[\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)\right] = \frac{A_1V_1}{g}\left(V_1\frac{A_1}{A_2} - V_1\right)$$

or

$$\left(\frac{p_2}{w} + z_2\right) - \left(\frac{p_1}{w} + z_1\right) = \frac{V_1^2}{g}m(1 - m) \qquad (3.21)$$

where m denotes the ratio (A_1/A_2) or (V_2/V_1).

The ratio of pipe areas which will give the maximum pressure rise can be found by rewriting Eq. (3.21) as

$$\Delta\left(\frac{p}{w} + z\right) = k(m - m^2)$$

Differentiating and equating to zero,

$$\frac{d\left(\frac{p}{w} + z\right)}{dm} = k(1 - 2m) = 0, \qquad m = \frac{1}{2}$$

The ratio of diameters for maximum pressure rise is then

$$\frac{d_1}{d_2} = \frac{1}{\sqrt{2}}$$

Considering mechanical energy only, the loss of energy can now be computed by applying Bernoulli's equation modified by the addition of a term representing this loss:

$$z_2 - z_1 + \frac{p_2 - p_1}{w} + \frac{V_2^2 - V_1^2}{2g} + h_L = 0$$

where h_L is the energy lost in the expansion. Substituting from Eq. (3.21),

$$\frac{V_1^2}{g}m(1 - m) + \frac{V_2^2 - V_1^2}{2g} + h_L = 0$$

Remembering that

$$V_2 = \frac{A_1}{A_2}V_1 = mV_1$$

$$h_L = \frac{V_1^2}{2g}(1 - m)^2 \tag{3.22}$$

Substituting for m in Eq. (3.22) its value, $m = V_2/V_1$,

$$h_L = \frac{V_1^2}{2g}\left(1 - \frac{V_2}{V_1}\right)^2 = \frac{V_1^2}{2g}\left(\frac{V_1 - V_2}{V_1}\right)^2$$

$$h_L = \frac{(V_1 - V_2)^2}{2g} \tag{3.23}$$

This is the Borda-Carnot equation for shock loss.

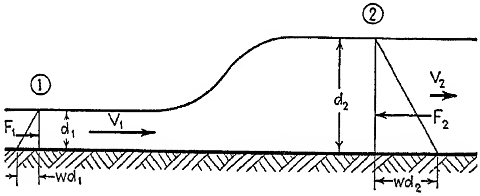

FIG. 36.

Equation (3.23) raises a very interesting point. Frictionless flow is being considered and yet a loss of mechanical energy is shown to occur under certain conditions. Experiment substantiates the theoretical result. The solution of this difficulty is that in computing the energy at point (2) only the translational velocity was considered whereas the dynamical conditions are such that a rotational velocity is generated. The energy of this rotation equals the apparent loss, which for all practical purposes is a real loss since the corresponding energy is no longer available for causing motion.

33. Hydraulic Jump in a Rectangular Horizontal Open Channel.—The hydraulic jump is an open-channel phenomenon similar to the sudden expansion in a pipe in that there is a rapid increase of flow area and a corresponding decrease of velocity accompanied by a loss of mechanical energy.

Figure 36 represents the elements which enter a discussion of the hydraulic jump. In practice, d_1 and V_1 are the known quantities and d_2 is desired. The jump can occur only when d_1 is less than a certain "critical depth." This critical depth will

be discussed in detail in a later chapter and will not be taken up at this point except to define it as $d_c = V_1^2/g$. In other words, the jump can occur only when $d_1 < V_1^2/g$.

Since there is a loss of energy of unknown magnitude, Bernoulli's equation cannot be applied, and it is necessary to resort to the momentum equation. Assume the fluid incompressible, the channel rectangular in section, the bottom of the channel horizontal, the flow at sections (1) and (2) parallel to the bottom, and no friction force acting. The resultant force acting to the right is

$$F_x = F_1 - F_2 = \frac{wd_1^2}{2} - \frac{wd_2^2}{2}$$

and by Eq. (3.17),

$$\frac{wd_1^2}{2} - \frac{wd_2^2}{2} = \frac{Qw}{g}(V_2 - V_1)$$

Considering a section of the channel 1 ft. wide,

$$d_1V_1 = d_2V_2 \text{ (cu. ft. per sec. per ft.)}$$

Substituting in the last equation and simplifying,

$$\frac{d_1^2 - d_2^2}{2} = \frac{d_1V_1^2}{gd_2}(d_1 - d_2)$$

Dividing both sides by $(d_1 - d_2)$,

$$\frac{d_1 + d_2}{2} = \frac{d_1V_1^2}{d_2g}$$

Solving for d_2,

$$d_2 = -\frac{d_1}{2} \pm \sqrt{\frac{d_1^2}{4} + \frac{2d_1V_1^2}{g}} \tag{3.24}$$

A more general form of the equation is

$$J = \tfrac{1}{2}(\sqrt{1 + 16r} - 1)$$

where $J = \dfrac{d_2}{d_1}$ and $r = \dfrac{V_1^2}{2gd_1}$

The loss of energy in the hydraulic jump can be evaluated by application of Bernoulli's equation, once d_2 is determined from the momentum equation:

$$h_L = \left(d_1 + \frac{V_1^2}{2g}\right) - \left(d_2 + \frac{V_2^2}{2g}\right)$$

34. The Pitot Tube.—The pitot tube is used for measuring fluid velocities, and depends for its action on the impact of the approaching fluid on the open end, which is directed into the stream. If we assume frictionless flow of a liquid in a large conduit of uniform cross section so that the pressures depend only on the velocities, the relation between velocity and pressure can be obtained from the momentum relations.

Considering a cylinder of water ahead of the pitot tube as illustrated in Fig. 37, there will be some point upstream from the tube at which the velocity in the cylinder is V_0, the velocity in the undisturbed pipe. Between this point and the end of the pitot tube, where the forward component of the velocity is zero, fluid is gradually passing through the walls of the imaginary tube. Confining attention to a small length of this tube, the net force acting toward the right is

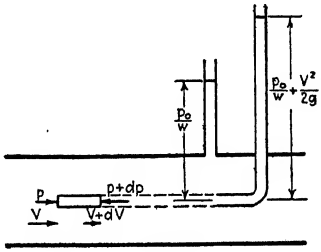

FIG. 37.

$$F = Ap - A(p + dp)$$

Letting m represent the mass per unit time, then

$$F = m_2V_2 - m_1V_1 - {}_1m_2\,{}_1V_2$$

where ${}_1m_2$ is the mass per unit time leaving the tube between sections. Referring to Fig. 37,

$${}_1V_2 = \frac{V + (V + dV)}{2}, \qquad V_1 = V, \qquad V_2 = V + dV$$

$${}_1m_2 = \frac{w}{g}A[V - (V + dV)]$$

$$m_1 = \frac{w}{g}AV, \qquad m_2 = \frac{w}{g}A(V + dV)$$

Making the substitutions,

$$Ap - A(p + dp) = \frac{w}{g}A(V + dV)(V + dV) - \frac{w}{g}AV^2 - \frac{w}{g}A\,dV\left(V + \frac{dV}{2}\right)$$

Simplifying, and discarding dV^2 as a differential of higher order,

$$-dp = \frac{w}{g} V\, dV$$

Integrating between the limits p_0 and p, and V_0 and 0, there results

$$\frac{p - p_0}{w} = \frac{V_0^2}{2g}$$

This may be solved for V_0:

$$V_0 = \sqrt{\frac{2g(p - p_0)}{w}} \tag{3.25}$$

There has been much discussion of the proper formula for the pitot tube and so brief a treatment may appear inadequate. The same results are obtained by writing Bernoulli's equation between a point remote from the tube and the stagnation point in the tube where $V = 0$.

Since the pitot tube will not be considered again, it may be well to mention the effect of compressibility on the reading. Starting from Eq. (3.5) and assuming adiabatic flow in the short reach immediately upstream from the impact opening, the final equation is

$$\frac{p - p_0}{w} = \frac{V_0^2}{2g}\left(1 + \frac{V_0^2}{4c^2} + \cdots\right) \tag{3.26}$$

where c is the velocity of sound for the conditions p_0 and ρ_0:

$$c = \sqrt{\frac{np_0}{\rho_0}}$$

For a velocity V_0 of 200 ft. per sec., at conditions corresponding to a velocity of sound of 1200 ft. per sec., the difference between the hydraulic and gas formulas [Eqs. (3.25) and (3.26)] is less than 1 per cent.

35. Borda Mouthpiece Discharging Liquid.—The Borda mouthpiece is simply a reentrant tube of such length that velocities along the face CD of the vessel in which it is placed are approximately zero, thus making the pressures on this face equal to those on AB, some distance back from the mouthpiece (see

Fig. 38). Under these conditions, the unbalanced force on the liquid between AB and CD is evidently hAw, where h is the distance from the water surface to the center of the tube, and A is the area of the tube, which is assumed to have a diameter small compared with h. Applying Eq. (3.17) to the flow of liquid from the vessel,

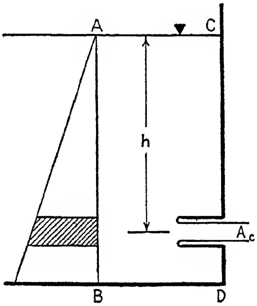

FIG. 38.

$$F = hAw = \frac{Qw}{g}V$$

By Bernoulli's equation, neglecting the small velocity in the vessel,

$$V = \sqrt{2gh}$$

and $Q = A_c\sqrt{2gh}$, where A_c is the area occupied by the stream of liquid when flowing at velocity V. Substituting,

$$hAw = \frac{A_c 2ghw}{g}$$

or

$$A = 2A_c$$

The area of the jet at its contracted section, where the pressure throughout is atmospheric (or equal to that at the top of the vessel), is half that of the tube. The discharge through the tube may be expressed as

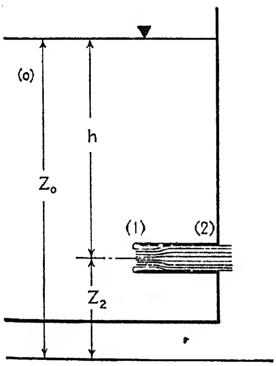

FIG. 39

$$Q = C_c A\sqrt{2gh}$$

where C_c is a coefficient applied to the area of the tube to give the area of flow. In this case, $C_c = 0.5$ and the theoretical equation for the discharge through a Borda mouthpiece is

$$Q = 0.5A\sqrt{2gh}$$

36. Borda Mouthpiece Running Full.—If the tube of the preceding discussion is sufficiently long compared with its diameter (three diameters or longer), the jet will expand and fill the tube. The

expansion is of the type already discussed. There is an energy loss equal to

$$h_L = \frac{V_2^2}{2g}\left(\frac{1}{m} - 1\right)^2$$

Writing Bernoulli's equation between points (0) and (2) of Fig. 39,

$$\frac{p_0}{w} + z_0 + \frac{V_0^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} + \frac{V_2^2}{2g}\left(\frac{1}{m} - 1\right)^2$$

But $p_0 = p_2$, $z_0 - z_2 = h$, $V_0 = 0$, and $m = 0.5$. Inserting these values, $h = V_2^2/g$, and $V_2 = \sqrt{gh}$. The discharge through the tube is

$$Q = A_2V_2 = A_2\sqrt{gh}$$

$$Q = \frac{1}{\sqrt{2}}A_2\sqrt{2gh}$$

where A_2 is the area of the tube. The ratio of discharges for the Borda mouthpiece running full and discharging freely is

$$\frac{Q_f}{Q_c} = \frac{\frac{1}{\sqrt{2}}A_2\sqrt{2gh}}{\frac{1}{2}A\sqrt{2gh}}$$

But $A_2 = A$ and therefore $Q_f/Q_c = \sqrt{2}$.

37. Sharp-edged Orifice.—The discussion of the Borda tube was based on the assumption that velocities along the surface parallel to the plane of the opening are small and that the corresponding pressure reductions are negligible. The unknown pressure distribution along the outside of the tube itself has no effect on the change of momentum and does not enter the summation of forces. Analysis of a sharp-edged orifice is essentially different as is illustrated by Fig. 40, where the downward force is proportional to the volume generated by

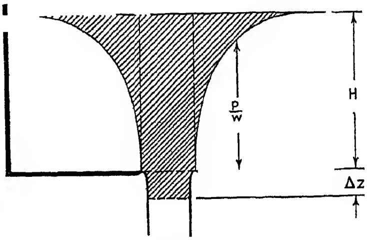

FIG. 40.

revolving the shaded area. Applying the momentum principle to obtain the coefficient of contraction,

$$C_c = \frac{g\left(A_pHw + \Delta W - \int_0^{A_p - A} p\, dA\right)}{wA2g(H + \Delta z)}$$

Here A_p is the total area of the horizontal plate and A is the area of the orifice. ΔW is the weight of the small amount of liquid between the plane of the orifice and the vena contracta. The quantities Δz and ΔW may be assumed as negligibly small. The only unknown quantity is the integral of the pressure over the surface of the plate, but evaluation of this term requires a knowledge

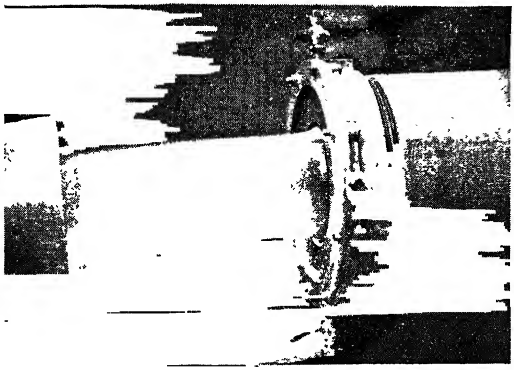

FIG. 41.

of the velocity distribution as can be shown by differentiating Bernoulli's equation and setting

$$dp = -\frac{w}{g} V\, dV$$

The methods presented thus far do not permit determination of the velocity distribution and the analysis cannot be carried farther except by making an arbitrary assumption. If the velocity is uniformly distributed over hemispheres concentric with the center of the orifice, the analysis gives $C_c = 0.535$ as compared with approximately 0.60 from experiment. The methods of hydrodynamics give 0.61 as the coefficient of contraction for a long, narrow slot.

If an orifice in a large plate is compared with one located in a pipeline having an area only slightly larger than the orifice, the

percentage of the total flow having a radial component is less for the pipeline orifice and the extent of the contraction should also be less than for the large plate. The magnitude of the contraction of the jet issuing from an orifice in a pipeline cannot be predicted theoretically, but it is a function of the ratio of the diameter of the orifice to that of the pipe. Figure 41 shows the jet from an orifice at the end of a pipeline.

38. Momentum Theory of Airfoil Action.—The principle of momentum may be employed to determine approximately the forces acting on an airfoil of finite span by making certain simplifying assumptions, of which the principal ones are that the cross-sectional area of the stream of fluid may be simply defined, and that all of the fluid is deflected through the same small angle

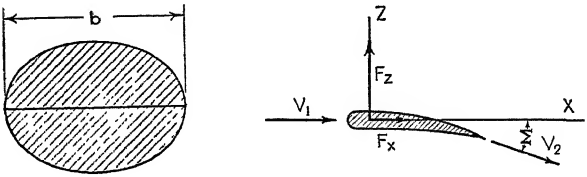

FIG. 42.

Σ, called the *angle of down-wash*. Selecting the coordinate axes as in Fig. 42, the force in the direction of the relative motion is called the *drag*, while that normal to it is called the *lift*.

The mass of air per unit of time deflected by the airfoil is assumed to be

$$M = K\rho b^2 V_1$$

where Kb^2 is the area of the stream cross section perpendicular to the velocity. As a simplification, we shall assume this area to be circular with a diameter equal to the airfoil span b (Fig. 42). Then

$$Kb^2 = \frac{\pi b^2}{4}$$

$$M = \rho\frac{\pi b^2}{4}V_1$$

Writing Eq. (3.18) for the lift force,

$$F_z = L = \rho\frac{\pi b^2}{4}(V_2 \sin \Sigma)V_1$$

Since $V_1 = V_2$ and the angle Σ is small, this becomes

$$L = \rho\frac{\pi b^2}{4}V^2\Sigma \tag{3.27}$$

Writing Eq. (3.17) for the drag force,

$$\begin{aligned} F_x = D &= \rho\frac{\pi b^2}{4}V_1^2(1 - \cos \Sigma) \\ &= \frac{2L}{\Sigma}\left(\sqrt{\frac{1 - \cos \Sigma}{2}}\right)^2 = \frac{2L}{\Sigma}\left(\sin \frac{\Sigma}{2}\right)^2 \\ &= \frac{2L}{\Sigma} \times \frac{\Sigma^2}{4} \end{aligned}$$

Therefore

$$D = L\frac{\Sigma}{2} \tag{3.28}$$

By a vector combination of the lift and drag, it is possible to determine the total force on the airfoil.

This treatment of airfoil lift is very elementary and is intended only as an illustration of the application of the momentum equations. For a more comprehensive treatment, students should consult standard works on aerodynamics.

39. General Statement of Momentum Equations for Steady Flow.—In applying the momentum equations to the preceding problems, each situation required careful examination in order that the forces, velocities, and pressures would be given the proper sign. The purpose of this section is to develop a routine procedure for writing these equations.

The foregoing particular applications of the momentum equations as applied to steady flow involved only a few relatively elementary principles which may be summarized as follows:

1. A region of the fluid system is isolated and treated as though it were a rigid body. What amounts to a free-body diagram is constructed for this region and a summation of forces is made for any selected direction ($\Sigma F_x = 0$).

2. The region is selected in such a manner that the pressures, velocities, and tractive forces are (*a*) known, or (*b*) have no effect on the motion. In a sudden expansion in a pipeline or in the hydraulic jump, the unknown distribution of pressure on the bottom acts at right angles to the direction considered and

the frictional force which acts in the direction of motion is too small to affect the result.

3. The exact flow phenomenon occurring within the region has no effect except in so far as it changes the component of the weight.

The forces to be considered in a more general treatment are the following:

PRESSURE FORCES.—The pressure acts perpendicularly on any surface. If the direction cosine of an inwardly directed perpendicular to a small area, dA, is α_a, the component of the pressure force over an area A_1 forming part of the boundary of the region is

$$\int_0^{A_1} p\alpha_a \, dA$$

MOMENTUM FORCES.—The method of applying the momentum equations is equivalent to assuming that all momentum carried by the fluid entering the region is annihilated, and that all momentum carried by the fluid leaving the region is created, at the boundary. The momentum force exerted on the fluid within the region is then

$$\int_0^{A_1} \rho(V\alpha_v)^2\alpha_a \, dA$$

where α_v is the direction cosine of the velocity.

TRACTIVE OR FRICTIONAL FORCES.—The internal friction of the fluid acts in such a manner as to resist local relative motion. The line of action of the frictional force is the same as that of the velocity but opposes the local velocity which may be in either direction. The fact that the direction of the frictional force depends upon the local velocity gradient makes a simple statement regarding this force difficult to formulate, although in particular cases its direction is seldom in doubt.

WEIGHT.—The weight of fluid within the region may have a component in the direction considered and this component must be included in the equilibrium equation. Other "body forces," such as centrifugal force, are treated in exactly the same manner.

$$W_x = W\alpha_w$$

where α_w is the direction cosine of the gravitational or other body force.

UNKNOWN FORCES.—The areas over which the pressure, momentum, and tractive forces are integrated need not be the entire area of the region. If part of the area is omitted, then an unknown reaction, R_x, representing the summation of pressure, momentum, and tractive forces over this area must be included.

The equilibrium equation for any arbitrarily assumed direction is

$$\int_0^{A_1} p\alpha_a \, dA + \int_0^{A_2} \rho(V\alpha_v)^2\alpha_a \, dA + F_x + W_x + R_x = 0$$

This equation is not so direct and concise as some statements of the momentum equations but it is believed to be in such form as to apply to any situation. Simpler statements of these equations generally assume that the pressure forces act in the same direction as the velocity and that they exist over the same areas as the flow. To illustrate the method of application, the equations will be used to solve problems.

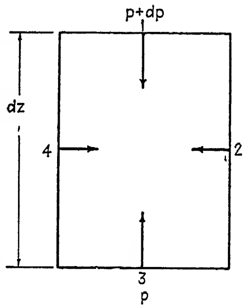

FIG. 43.

1. *Hydrostatic Equilibrium.*—Consider a vertical column of fluid of *unit area* and height *dz*. The fluid is stationary and, consequently,

$$F_z = 0 = R_z$$

$$\int_0^A (V\alpha_v)^2\alpha_a \, dA = 0$$

The direction cosines of the vertical area are zero but $\alpha_{a1} = -1$, and $\alpha_{a3} = +1$.

The pressure force is

$$\int_0^A p\alpha_a \, dA = p\,1\,(+1) + (p + dp)\,1\,(-1)$$
$$= -dp$$

The direction cosine of the gravitational force is -1 and the component of the weight is

$$-\rho g\,1\,dz$$

The equilibrium equation is $-dp - \rho g \, dz = 0$,

$$dp = -\rho g \, dz$$

2. *Hydraulic Jump in a Rectangular Horizontal Channel.*—The region considered extends from (1) to (2). Since the atmospheric pressure is merely an additive constant which acts over all areas concerned, it may be taken as zero. The bottom is parallel to the direction of motion and the pressures acting on it do not enter the problem. The direction cosines are

$$\alpha_{a1} = 1, \qquad \alpha_{v1} = 1$$
$$\alpha_{a2} = -1, \qquad \alpha_{v2} = -1$$

$$\int_0^A p\alpha_a\, dA = +1\int_0^{D_1} wz\, dz + (-1)\int_0^{D_2} wz\, dz$$
$$= \frac{w}{2}(d_1^2 - d_2^2)$$

$$\rho = \text{const.} = \frac{w}{g}$$

$$\frac{w}{g}\int_0^{A_2} (V\alpha_v)^2\alpha_a\, dA = \frac{w}{g}[(+1V_1)^2(+1)d_1 + (-1V_2)^2(-1)d_2]$$

The equilibrium equation is

$$\frac{w}{2}(d_1^2 - d_2^2) + \frac{w}{g}(d_1V_1^2 - d_2V_2^2) + F_s = 0$$

where F_s is the frictional resistance along the bottom. The length of the jump is not great and F_s is small as compared with the pressure and momentum forces involved. The example will not be carried further since the hydraulic jump has been considered previously.

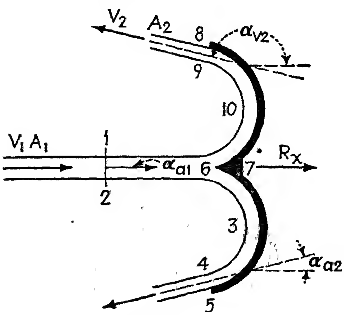

FIG. 44.

3. *Force Exerted on an Impulse Turbine Bucket.*—Consider a bucket which has been so designed that all water striking it is deflected at the angle $\alpha_{v2} = 180$ deg. To obtain the unknown force exerted on the fluid along the boundary surface 5-6-8-7, integrate the pressure and momentum forces over the surface 8-9-10-1-2-3-4-5. Remembering that the inwardly directed normals to the areas are to be used,

$$\alpha_{a1} = +1, \qquad \alpha_{a2} = +1$$
$$\alpha_{v1} = +1, \qquad \alpha_{v2} = -1$$
$$\rho = \text{const.} = \frac{w}{g}$$
$$p_1 = p_2 = p = 0, \qquad \int p\alpha_a \, dA = 0$$
$$\rho\int_0^A (V\alpha_v)^2\alpha_a \, dA = \rho(V_1^2A_1 + V_2^2A_2) = Q\rho(V_1 + V_2)$$
$$\frac{w}{g}Q(V_1 + V_2) + R_x = 0$$

In a numerical problem, R_x would be found to be negative, indicating that it acts in a direction opposite to that assumed. It is the summation of the components of both the tractive and pressure forces acting on the face of the bucket.

If the same method is applied to obtain the components of an unknown reaction parallel to three mutually perpendicular axes, the magnitude of the resultant force would be

$$R = (R_x^2 + R_y^2 + R_z^2)^{\frac{1}{2}}$$

but the line of action of this resultant would be unknown and must be determined by considering the moments of the forces.

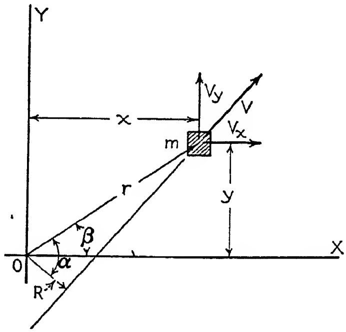

Fig. 45.

40. Relationship between Torque and Time Rate of Change of Moment of Momentum.—A very important principle in the analysis of centrifugal pumps, blowers, and turbines is that the torque exerted about any axis equals the time rate of change of moment of momentum about that axis. The principle will first be developed for the movement of a single particle of mass m and will then be modified so as to be applicable to the steady flow of a fluid.

Referring to Fig. 45, a single particle of mass m is located at the point (x,y) and is moving with a velocity V relative to the origin of coordinates. The moment of momentum about the axis is VRm and is, in terms of the components,

$$VRm = V_y xm - V_x ym = \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)m$$

The time rate of change of moment of momentum about the axis is obtained by differentiating with respect to time and is

$$m\frac{d(VR)}{dt} = \left(x\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{dx}{dt} - \frac{dx}{dt}\frac{dy}{dt} - y\frac{d^2x}{dt^2}\right)m$$
$$= (xa_y - ya_x)m$$

where a_y and a_x are the components of acceleration. The components of the force acting are $F_x = a_xm$ and $F_y = a_ym$. The equation reduces to

$$m\frac{d(VR)}{dt} = xF_y - yF_x = T$$

where T is the torque or moment about the axis. But $R = r \cos \alpha$ and

$$T = m\frac{d(Vr \cos \alpha)}{dt} \tag{3.29}$$

Equation (3.29) applies to the movement of a single unit of mass. In flow problems it is difficult to trace the history of any particular mass and for steady flow it is convenient to modify this equation in the following manner: Considering Fig. 46, a slug of fluid initially occupying the position $AABB$ advances during a short time interval dt to the position $A'A'B'B'$. The mass between $A'A'$ and BB remains unchanged in moment of momentum and the net change is obtained by subtracting the moment of momentum of the shaded portions. The mass of each end element is

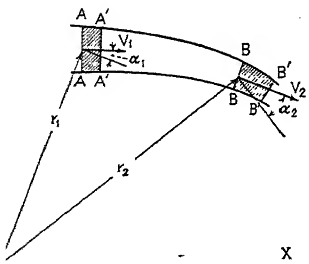

Fig. 46.

$$A_1\rho_1V_1\, dt = A_2\rho_2V_2\, dt = \frac{W}{g}dt$$

where W is the weight of fluid per unit time. The change in moment of momentum is

$$\frac{W}{g}(r_2V_2 \cos \alpha_2 - r_1V_1 \cos \alpha_1)dt$$

The rate of change with time is evidently obtained by dividing through by dt, and therefore

$$T = \frac{W}{g}(r_2V_2 \cos \alpha_2 - r_1V_1 \cos \alpha_1) \tag{3.30}$$

The mass m in Eq. (3.29) has been replaced by the rate at which mass passes any section.

Equation (3.30) forms the basis for a number of important developments, notably for the theoretical treatment of centrifugal pumps and turbines. Another group of problems involves the condition that $T = 0$ in which event, $r_1V_1 \cos \alpha_1 = r_2V_2 \cos \alpha_2$. For example, if a tank is supplied at the side and discharges at the center, an initial tangential component of velocity would increase inversely as the radius as the fluid moved towards the center. On the free surface of a liquid this increase in velocity of rotation may be accompanied by a considerable depression of the surface.

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Problems

1. A circular orifice 1 sq. in. in area is made in the vertical side of a large tank. If the jet falls vertically through $1\frac{5}{8}$ ft. while moving horizontally through 5 ft., at the same time discharging 16 gal. of water per minute, determine the horizontal force on the tank. *Ans.* 1.072 lb.

2. Water flows through a horizontal circular pipe of 4-in. diameter which is reduced to a 2-in. diameter. A water-mercury manometer shows a differential reading of 15 in. of mercury when connected between the 4- and 2-in. sections. Assuming a perfect fluid, compute the rate of flow in the pipe. *Ans.* 0.717 cu. ft. per sec.

3. Water from a large open storage tank is discharged through a pipeline consisting of (*a*) 5-in., (*b*) 6-in., (*c*) 4-in., and (*d*) 2-in. sections of pipe. If the center lines of all pipes are horizontal and placed at the same elevation, compute the rate of discharge from the 2-in. line (against atmospheric pressure) when the free surface in the storage tank is maintained at an elevation of 35 ft. above the pipe center line. Compute the velocity and pressure in each pipe section. Neglect all friction, sudden contraction, and expansion losses.

4. The gage pressure in a horizontal 10-in. pipe is 15 lb. per sq. in. and the velocity is 5 ft. per sec. At some distance farther, the pipe is again level, 6 in. in diameter, and 5 ft. higher than the first section. What is the gage pressure in pounds per square inch? *Ans.* 11.7 lb. per sq. in.

5. In Prob. 4, what must be the diameter of the pipe at the second section if the gage pressure there is to be 5 lb. per sq. in.? *Ans.* 3.8 in.

6. From the bottom of an open tank of water 30 ft. deep is a vertical pipe discharging freely into the air at a point whose distance is Z below the bottom of the tank. Within the pipe and at a point which is 32 ft. below the surface of the water, the pressure is 10 lb. per sq. in. less than atmospheric. Neglecting all losses, determine the length of the pipe.

7. A 24-in. diameter pipeline discharges water through a 2-in. diameter orifice. A pressure gage indicates 100 lb. per sq. in. in the pipe when the discharge is against atmospheric pressure. Compute the theoretical discharge (*a*) neglecting velocity in large pipe and (*b*) considering all velocities.
Ans. (*a*) 121.9 ft. per sec.; (*b*) 121.9+ ft. per sec.

8. A reservoir discharges through a 2-in. diameter opening into a conical pipe expanding to a 4-in. diameter. What is the discharge in cubic feet per second when the reservoir level is 20 ft. above the opening?
Ans. 3.1 cu. ft. per sec.

9. A venturi meter in a 4-in. pipeline has a 2-in. diameter throat. What is the differential head in feet of water when 1.0 cu. ft. of water per second is flowing? *Ans.* 30.7 ft.

10. A horizontal tube converging from an 8-in. to a 4-in. diameter, carries water with a velocity of 5 ft. per sec. at the larger section and the total energy content at that point is 6 ft.-lb. per lb. Find the pressure in pounds per square inch at the 4-in. section. *Ans.* −0.091 lb. per sq. in.

11. A tank discharges water into another tank with free surface 4 ft. below the first through a horizontal conical diffuser, whose throat is 2 in. in diameter. The axis of the cone is 10 ft. below the surface of the water in the first tank. Compute the diameter D_0 of the large end of the cone. and the rate of discharge Q for perfect vacuum conditions at the throat. Neglect all losses.

12. A venturi meter in a 5-in. line carrying water has a 3-in. throat. A water-carbon-tetrachloride differential manometer connected to the upstream pipe and the throat shows a deflection of 3.87 ft. If the sp. gr. of CCl_4 is 1.59, what is the rate of flow in cubic feet per second?

13. A venturi meter for a 6-in. pipeline has a 3-in. throat diameter and is placed with its axis horizontal. At a discharge of 3.0 cu. ft. per sec., what is the differential pressure in feet of water? What is the differential deflection if a mercury manometer is used?

Ans. (*a*) 0.107 lb. per sq. in.; (*b*) 0.994 ft.

14. Air (assume dry) flows through a venturi tube having a 9-in. diameter throat. If the conditions upstream from the tube are pressure 29.75 in. of mercury, temperature 80 deg. Fahr., and the area large enough so that the initial velocity can be neglected, what is the air velocity at the throat (*a*) neglecting compressibility and (*b*) considering compressibility? The differential pressure is 3 in. of water.

Ans. (*a*) 117.1 ft. per sec.; (*b*) 118.4 ft. per sec.

15. Air discharges into the atmosphere (14.7 lb. per sq. in.) from a very large tank through a $\frac{1}{4}$-in. nozzle. The temperature in the tank is 140 deg. Fahr. and the pressure 50 lb. per sq. in. gage. What is the rate of discharge in pounds per second? Assume discharge coefficient is unity.

Ans. 0.0685 lb. per sec.

16. The quantity of oxygen flowing in a pipeline is to be determined with the aid of an orifice. At a point ahead of the orifice the gas is under a pressure of 100 lb. per sq. in. abs. and a temperature of 100 deg. Fahr. What is the greatest downstream pressure for maximum velocity through the orifice?

17. A nozzle on the end of a 6-in. pipe discharges a 2-in. diameter jet of water. In the pipe, the pressure is 55 lb. per sq. in. gage and the velocity 10 ft. per sec. The jet is discharged into the air. (*a*) What is the resultant force acting on the water within the nozzle? (*b*) What is the axial component of the force exerted on the nozzle? *Ans.* (*a*) 304 lb.; (*b*) 1251 lb.

18. A horizontal jet issuing from an orifice 1 in. in diameter ($C_c = 1.0$) discharges 2 gal. of water per second and impinges on a curved surface which deflects the stream through a total angle of 150 deg. Determine the force exerted on the surface. *Ans.* 48.9 lb.

19. A horizontal reducing elbow, 8 by 6 in. carries 4.35 cu. ft. per sec. of glycerin, sp. gr. 1.26. The pressure on the 8-in. end is 25 lb. per sq. in. What is the total resultant force tending to move the elbow?

20. A tank containing mercury to a depth of 8.0 ft. discharges through a circular opening whose diameter is 6 in. and whose center is 12 in. above the bottom of the tank. Assume that the fluid is not contracted as it passes the opening. What is the force tending to move the tank?

Ans. 2310 lb.

21. An 8-in. 45-deg. elbow carries 4 cu. ft. per sec. of water at a pressure of 50 lb. per sq. in. Compute the total resultant force tending to move the elbow.

22. Oil of sp. gr. 0.84 is pumped through a 4-in. diameter pipeline into a 2-in. diameter line through a 45-deg. reducing elbow. In the larger pipe, the pressure is 100 lb. per sq. in. abs. and the velocity is 8 ft. per sec. Determine the magnitude and direction of the resultant force on the elbow if the piping system is at a constant elevation. Neglect all losses.

Ans. (*a*) 1058 lb. (*b*) 12 deg. 55 min. with the pipe.

23. A 5-in. pipe discharges into a 15-in. pipe. What is the difference in pressure between the two pipes if the water velocity in the smaller is 9 ft. per sec.? What is the head lost in the sudden expansion?

24. Oil of sp. gr. 0.80 flows in a 3-in. circular pipeline. A sudden expansion takes place into a second pipe of such diameter that the maximum pressure rise is obtained. If 3.142 cu. ft. per sec. flows in the pipeline, find (*a*) the loss of energy in the sudden expansion, and (*b*) the differential height h of an oil-mercury manometer connected between the two pipes.

Ans. (*a*) 15.9 ft. of oil; (*b*) 23.8 in.

25. If a 2-in. pipeline carrying 1.0 cu. ft. of water per second at 80 lb. per sq. in. suddenly discharges into a 4-in. pipe, what is the pressure in the 4-in. pipe in pounds per square inch? What is the loss of energy in feet of water? *Ans.* 1990 lb.

26. Water flows over a dam and flows horizontally at the bottom with a depth of 1.5 ft. and a velocity of 40 ft. per sec. If the hydraulic jump occurs, what is the depth of water below it? What is the rate of energy dissipation in horsepower per foot of width?

Ans. (*a*) 11.48 ft.; (*b*) 97.3 hp.

27. Water discharges from an 8-in. line through a 2-in. pipe against atmospheric pressure. A pressure gage on the 8-in. line indicates 180 lb. per sq. in. What would be the height of a water column connected with a pitot tube in the large line? Compute the rate of discharge in cubic feet per second. *Ans.* (*a*) 416.6 ft.; (*b*) 3.58 cu. ft. per sec.

28. A differential water gage when connected to a pitot tube and a pipe static pressure connection reads 25 in. of water, the pitot tube being placed at the center of an 8-in. air line. If a pressure gage on the pipe reads 75 lb. per sq. in. gage and a thermometer reads 140 deg. Fahr., compute the velocity of air at the center of the 8-in. pipe, (*a*) neglecting compressibility of air and (*b*) considering compressibility of air. Neglect elevation of water gage above center line of the pipe and assume the air is dry.

Ans. (*a*) 144.0 ft. per sec.; (*b*) 143.7 ft. per sec.

29. The velocity of water in a pipe is measured by means of a pitot tube, a static pressure connection, and a differential manometer containing water and mercury. If the manometer differential reading is 1.67 ft., what is the velocity of flow? *Ans.* 36.8 ft. per sec.

30. In measuring the velocity of air with a pitot tube, what is the approximate maximum velocity for which the error is less than 1.5 per cent if the compressibility is neglected? *Ans.* 416 ft. per sec.

31. The velocity of oil in a pipe is measured by a pitot tube. A mercury-oil differential manometer reads 5 in. when connected to the pitot tube and a static pressure opening in the pipe. What is the velocity of the oil if its sp. gr. is 0.85? *Ans.* 20.1 ft. per sec.

32. Air discharges from a large tank through a Borda mouthpiece. If the downstream pressure is greater than the critical pressure and a = area of tube, ρ = density of air in tank, V = velocity at vena contracta, obtain the expression for C_c, the coefficient of contraction.

33. Water discharges from a pipe into a tank which is placed on scales. The discharge velocity is 5 ft. per sec. and the cross-sectional area of the tank is 3 sq. ft. Compute the difference in weight indicated by the scales when the tank fills from a depth of 1 ft. to 5 ft. The end of the discharge pipe is 8 ft. above the bottom of the tank. *Ans.* 745.6 lb.

34. A large tank has a 4-in. circular opening in one side 18 in. above the bottom. A 3-in. length of pipe projects into the tank. If the depth of water in the tank is 40 ft., what is the rate of discharge? What is the unbalanced force tending to move the tank?

Ans. (*a*) 2.18 cu. ft. per sec.; (*b*) 211 lb.

35. In Prob. 34, if the pipe be extended to a length of 2 ft. what is the discharge, neglecting friction losses? *Ans.* 3.08 cu. ft. per sec.

36. An open tank containing mercury with the level maintained at 9 ft. above the center line of a Borda mouthpiece (length greater than four diameters) discharges into air maintained at a pressure of 2 ft. of mercury above atmospheric. If the tube is 6 in. in diameter with its center 12 in. above the bottom of the tank, find the force tending to move the tank.

Ans. 1162 lb.

37. Oil of sp. gr. 0.90 is maintained in a tank at 10 ft. above the center line of a nozzle in the side of the tank. The discharged circular jet ($C_c = 1$) strikes a large flat plate which covers a short tube of diameter equal to the jet diameter. To what height h can water be added to a second tank connected to the short tube before the plate will move and allow water to be discharged from the second tank? *Ans.* 18 ft.

38. A jet of water having a diameter of 1.0 in. and velocity of 100 ft. per sec. strikes a cup and is deflected through 120 deg. Find the magnitude and direction of the resultant force on the cup.

39. If the cup of Prob. 38 travels away from the nozzle with a velocity of 50 ft. per sec. and the deflection is the same, what are the magnitude and direction of the resultant?

40. Water is supplied to a locomotive tender by means of a scoop dipping into a track trough. If the entrance to the tank is 10 ft. above the trough and the discharge diameter is 2 in., what are the velocity and rate of discharge

when the locomotive is moving at 40 mi. per hr.? What is the train speed at which flow starts? Assume frictionless flow.

Ans. (*a*) 53.1 ft. per sec.; (*b*) 1.16 cu. ft. per sec.; (*c*) 17.25 mi. per hr.

41. If the scoop of Prob. 40 discharges horizontally toward the back of the tender, what is the total horizontal force on the scoop? Compute the total horizontal force on the scoop if it discharges horizontally toward the front of the tender. *Ans.* (*a*) 13.3 lb.; (*b*) 246 lb.

CHAPTER IV

VISCOSITY, TURBULENCE, AND FRICTION

The preceding chapter was devoted to the motions of, and external forces acting on, frictionless fluids. It was assumed that the fluid could not transmit tangential stresses, and internal motions were entirely neglected. However, all real fluids exhibit a physical property known as *viscosity* which gives rise to tangential stresses, and which is associated with the degradation of mechanical energy into heat. The equations developed for the so-called *perfect* fluid must be modified in practice to account for the effect of viscosity, and it is the purpose of this chapter to discuss the basic principles involved and to utilize these principles in devising proper corrections.

More specifically, this chapter will deal with friction losses in pipes and channels and the corresponding modifications in the Bernoulli energy equation. In the general form this equation becomes

$$\int_2^1 \frac{dp}{w} + (z_1 - z_2) + \frac{V_1^2 - V_2^2}{2g} = \Delta E$$

where ΔE is the change in mechanical energy per unit weight between two points in a flow system. For liquids, this equation reduces to

$$\frac{p_1 - p_2}{w} + (z_1 - z_2) + \frac{V_1^2 - V_2^2}{2g} = h_L$$

Here, h_L is commonly referred to as the *loss of head.* It is the decrease in available mechanical energy per unit weight of fluid and is positive if the flow is from point (1) to point (2). In general, the decrease of mechanical energy appears as heat and causes an expansion of the fluid but this effect is appreciable only in gases. Whether ΔE should be referred to as a loss really depends upon the nature of the system. For example, in a heating system the loss of mechanical energy is available for

useful purposes. Except where otherwise specified, it will be assumed that the heat-transfer conditions are such that the flow is isothermal and that the heating by friction has a negligible effect.

41. Viscosity.—The earliest definition of viscosity is that contained in Newton's "Principia." Quoting from Hatschek, Newton's definition is: "The resistance which arises from lack of slipperiness, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another." This definition may be illustrated by considering the flow between two flat plates moving parallel to each other with constant pressure at all points between them ($dp/dx = 0$). The space between the plates is completely filled with fluid which adheres to both plates (relative velocity at each plate zero). The force necessary to move the plates is the same as the shearing force at any position z, since the velocity of each lamina of fluid is constant and the fluid must possess the property of resisting motion if the assumed conditions are to exist. The coefficient of absolute viscosity μ is defined by means of the relationship

$$F = \mu A \frac{V}{h} \tag{4.1}$$

or

$$\frac{F}{A} = \tau = \mu \frac{V}{h}$$

Here τ is the shearing force per unit area. If the coefficient μ is found experimentally to depend only upon the nature of the liquid and not upon the values of F, A, V, and h, it is termed a *physical property* of the fluid. Conversely, Eq. (4.1) may be taken as the definition of a fluid; *i.e.* any substance which has a value of μ independent of the flow conditions is a true fluid.

The quantities V and h are of any magnitude below certain critical values which will be discussed later and may be considered as differentials. The equation for the shearing force per unit area at any elevation z is then,

$$\tau_z = \mu \left(\frac{dV}{dz} \right)_z \tag{4.2}$$

Equation (4.2) may be transformed slightly to give a different representation of the viscous force and one which may have more

significance. Referring to Fig. 47, the tangent of the angle swept through in time Δt by a row of particles initially perpendicular to the plates is

$$\tan \Delta\alpha = \frac{V\Delta t}{z}$$

As

$$\Delta t \to 0, \qquad \tan \Delta\alpha \to \Delta\alpha = d\alpha,$$

and

$$\frac{d\alpha}{dt} = \frac{V}{z} = \frac{dV}{dz}$$

$$\tau = \mu\frac{d\alpha}{dt}$$

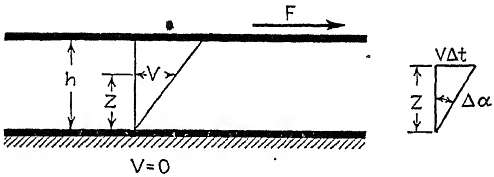

FIG. 47.

Here $d\alpha/dt$ is the angular velocity of a line of fluid particles perpendicular to the velocity V but not that of any other line. The viscous resistance is seen to be related to the rate of distortion of the fluid elements.

The dimensions of absolute viscosity are obtained directly from Eq. (4.2) as follows:

$$[\mu] = \left[\frac{\tau}{\frac{dV}{dz}}\right] = \left(\frac{\text{force per unit area}}{\text{velocity gradient}}\right)$$
$$= FTL^{-2} = ML^{-1}T^{-1}$$

In the English system, the units are pound-seconds per foot square or slugs per foot-second.

So many different methods of expressing viscosity are in vogue that it is difficult to use tabulated data. The difficulty results largely from the two different systems of expressing weights and masses. However, in nearly all flow problems, the ratio of the absolute viscosity to the mass per unit volume is the important

variable. This quantity is known as the *kinematic viscosity* and has the dimensions

$$[\nu] = \left[\frac{\mu}{\rho}\right] = L^2T^{-1}$$

In addition to being the quantity most frequently used, the kinematic viscosity does not include the dimensions of mass or force. It is convenient to remember that the kinematic viscosity of water at normal air temperatures is about 1.0×10^{-5} ft.^{2} per sec. and that the density is 1.94 slugs per cu. ft. These two values may be used to check tables of data since their product is the absolute viscosity.

42. The Nature of Viscous Resistance.—In the kinetic theory of gases, viscous resistance is shown to arise from the transfer of momentum between layers of gas moving at different velocities. The molecules are assumed to be in random motion as regards both direction and velocity, but the average effect is the same as though one-third of the molecules moved parallel to each of three mutually perpendicular axes with a velocity which is the square root of the average of the squares of the individual velocities. Although this conception is applicable only to gases, in which the distance between molecules is relatively great, it provides an approximate picture of the origin of viscous resistance in liquids, and it also provides a basis for the theory of turbulent flow to be developed later. The simplified theory outlined here is that given by Bingham.

Referring to Fig. 48, steady flow is assumed to occur in such a manner that the velocity diagram is the same in all planes parallel to the sketch. The significance of the symbols is as follows:

N = number of molecules in a unit volume.
c = average molecular velocity.
m = mass per molecule.
L = mean free path between molecular collisions.
ρ = density.
V = velocity of the center of mass at z.

At the moment of collision of molecules at points along the plane AB, the average forward velocity is V. Those molecules which move upward pass into a region of higher average velocity, while those which move downward pass into a region of lower velocity.

The net effect is such that a force in the X-direction above the plane AB is necessary to maintain steady motion, and hence the momentum transfer is equivalent to the effect of a tractive force applied along that plane.

Since one-third of the molecules may be considered as moving along paths perpendicular to AB, and since it is as likely that they will be moving upward as downward, the mass per second moving downward across unit area of the plane AB is $\frac{1}{6}Nmc$. The average X-component of momentum transferred downward per second across the plane AB by these particles is

$$\frac{1}{6}Nmc\left(V + L\frac{dV}{dz}\right)$$

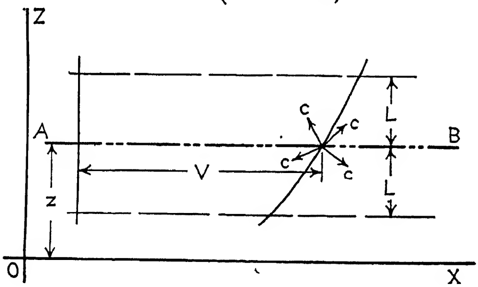

FIG. 48.

Similarly for upward moving particles, the momentum transferred per second in the X-direction is

$$\frac{1}{6}Nmc\left(V - L\frac{dV}{dz}\right)$$

Subtracting the upward transfer of momentum from the downward transfer gives as the net rate of transfer per unit area per second,

$$\frac{1}{3}NmcL\frac{dV}{dz}$$

This quantity is the net loss of momentum downward across plane AB per unit area per second and represents a tendency to retard the fluid above AB and accelerate that below. It is therefore equivalent to a tractive force and we may write

$$\tau = \frac{1}{3}\rho cL\frac{dV}{dz} \tag{4.3}$$

Comparing Eq. (4.3) with Eq. (4.2),

$$\mu = \tfrac{1}{3}NmcL \tag{4.4}$$

The density of a gas varies nearly as the pressure while the mean free path is inversely proportional to the pressure. The molecular velocity is a function only of the temperature ($T \propto c^2$). These facts lead to the conclusion that the absolute viscosity of a gas should be independent of pressure but proportional to the square root of the absolute temperature.

$$\mu = (K_1p)\left(\frac{K_2}{p}\right)\sqrt{K_3T} \propto \sqrt{T}$$

This conclusion agrees fairly well with measured values in the range of conditions in which the assumptions of this theory are valid.

It should be noted that this theory does not apply directly to liquids, but the general concept of viscosity as resulting from a transfer of momentum is probably correct for liquids as well as for gases. However, the molecules forming a liquid are more closely packed than in gases and there may occur interlocking or sliding in addition to the transfer of momentum by random motion.

43. Dissipation of Mechanical Energy.—Viscous flow involves the conversion of mechanical energy into heat. The rate of this conversion per unit volume in *two-dimensional flow* problems is obtained as follows:

τV = power per unit area due to shearing force at any point z

$\dfrac{d}{dz}(\tau V)dx\ dy\ dz$ = power expended on volume element of area $dx\ dy$ and thickness dz

Considering the simplest case of steady flow with $dp/dx = 0$, all of the power represented by the gradient of τV is expended on work of deformation. Under the assumed conditions, $d\tau/dz = 0$ and the power expended in deforming the volume element $dx\ dy\ dz$ is $\tau\dfrac{dV}{dz}dx\ dy\ dz$. The power per unit volume at point z is then

$$\tau\frac{dV}{dz} = \mu\left(\frac{dV}{dz}\right)^2$$
$$= \mu\left(\frac{d\alpha}{dt}\right)^2$$

A similar expression can be obtained for the rate of dissipation of mechanical energy in three-dimensional flow.

44. Isothermal Laminar Flow in Circular Tubes. (Poiseuille's Law.)—Considering Fig. 49, flow occurs vertically upward in a circular tube of radius R. Conditions are assumed to be such that the flow will be laminar and the problem is to predict the velocity distribution, discharge, and velocity head for known values of p_1, p_2, and L. It is also assumed that the flow is isothermal, which is equivalent to the assumption that μ = const.

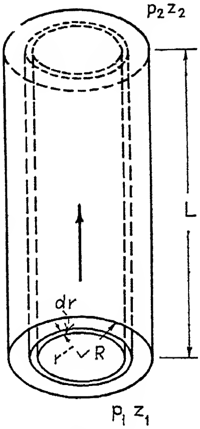

FIG. 49.

The forces acting on the cylinder of fluid of radius r concentric with the tube are the pressure difference, the weight, and the viscous resistance. The equilibrium equation is then

$$(p_1 - p_2)\pi r^2 + w(z_1 - z_2)\pi r^2 + 2\pi r L\frac{dV}{dr} = 0 \qquad (4.5)$$

Dividing through by $(Lw\pi r^2)$ and collecting te

$$\frac{1}{L}\left[\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)\right] + \frac{2\nu}{gr}\frac{dV}{dr} = 0$$

The first terms in the equation represent the slope of the hydraulic grade line between the sections. Indicating this quantity by h' and solving for the velocity gradient,

$$\frac{dV}{dr} = -\frac{rgh'}{2\nu}$$

Integration gives

$$V = -\frac{gh'}{2\nu}\cdot\frac{r^2}{2} + C$$

The boundary conditions are that at $r = R$, $V = 0$, and consequently $C = gh'R^2/4\nu$. The complete velocity equation is

$$V = \frac{gh'}{4\nu}(R^2 - \qquad (4.6)$$

The maximum velocity occurs where $r = 0$ and its value is $gh'R^2/4\nu$. If the tube is horizontal, Eq. (4.6) takes on a more familiar form:

$$V = \frac{p'}{4\mu}(R^2 - r^2)$$

Here, p' is the pressure gradient and μ is the absolute viscosity.

In writing Eq. (4.5), it was tacitly assumed that the velocity gradient had a constant value at all points on the circle of radius r. From symmetry such a condition is to be expected in a long tube but the velocity distribution may be considerably distorted near the entrance.

The *discharge* per unit time is obtained by integrating under the velocity curve:

$$\begin{aligned} Q &= \int_0^R 2\pi r V\,dr \\ &= \frac{\pi p' R^4}{8\mu} \qquad (4.7) \end{aligned}$$

Equation (4.7) is that used by Poiseuille in the measurement of viscosity.

The kinetic energy of the fluid passing any section in unit time is

$$\begin{aligned} KE &= \int_0^R 2\pi r(\rho V)\frac{V^2}{2}dr \\ \int_0^R rV^3\,dr &= \frac{p'^3}{64\mu^3}\int_0^R (R^6 r - 3R^4 r^3 + 3R^2 r^5 - r^7)dr \\ &= \frac{p'^3 R^8}{512\mu^3} \\ KE &= \frac{\pi\rho p'^3 R^8}{512\mu^3} \end{aligned}$$

The average kinetic energy per unit weight, or the average velocity head, is

$$h_v = \frac{\dfrac{\pi\rho p'^3 R^8}{512\mu^3}}{\dfrac{\pi\rho g p' R^4}{8\mu}} = \frac{p'^2 R^4}{64 g\mu^2} \tag{4.8}$$

In computing the average velocity head, it is convenient to use the mean velocity as determined from the rate of discharge divided by the area. The square of the mean velocity is less than the weighted average of the squares of the individual velocities and a corrective coefficient, α, must be introduced:

$$h_v = \alpha\frac{V_m{}^2}{2g} = \alpha\frac{Q^2}{2g\pi^2R^4}$$

Substituting for h_v from Eq. (4.8) and for Q from Eq. (4.7), gives $\alpha = 2$. This coefficient is the same regardless of the mean velocity, so long as the flow is laminar. It accounts for the major portion of the velocity-head correction in the equations for the capillary-tube viscometer.

45. Criteria of Similarity of Flow.—It is a well-known fact that most of the flows dealt with in engineering are not laminar and do not follow the equations cited in the preceding sections. Hence other means of predicting the friction losses are necessary. The number of different fluids for which friction computations must be made is so great that a generalized representation for turbulent flow, such as Poiseuille's law for laminar flow, is very desirable. Such a generalization was provided by Osborne Reynolds in the form of a criterion of similarity known as the *Reynolds number*.

In the field of fluid mechanics there are a number of criteria which are used to generalize experimental results on the principle that when these criteria have equal values in two different fluid systems, these systems will exhibit similar characteristics. In studying the resistance of ship hulls to motion through water, a criterion developed by Froude is used to indicate the conditions for similarity of the wave pattern and equality of the coefficient of resistance. Other criteria are used in generalizing data on forced and free convection of heat, the formation of capillary waves, and so forth. Each one of these criteria indicates the conditions for similarity of behavior under the influence of certain physical phenomena, such as, for example, the gravita-

tional force of the earth, or surface tension, and they do not apply when these phenomena exert a negligible influence.

Inertia and viscosity are the basic phenomena involved in the frictional resistance of fluids, and the criterion of similarity must be such as to indicate a similar relationship between them. In deriving the Reynolds criterion, two essentially different methods will be employed, one of which considers directly the dynamical relationships involved, while the other is based upon dimensional analysis.

46. Dynamical Similarity.—Dynamical similarity of flow in the ordinary sense presupposes geometrical similarity of the solid boundaries, including the roughness, and sets the conditions

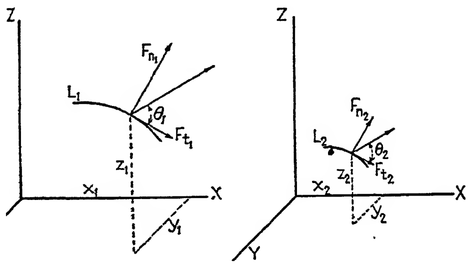

FIG. 50.

for geometrical similarity of the paths traced out by the fluid. Referring to Fig. 50, the paths L_1 and L_2 are geometrically similar if the coordinates x_1,y_1, and z_1 of points along the path L_1 can be used to construct the path L_2 by multiplying each coordinate by the same scale factor, which will be indicated here by b_l. The coordinate axes are of course assumed to be similarly oriented in relation to the geometrically similar boundaries. For geometrical similarity, all corresponding linear dimensions, such as the radii of curvature of the paths of the fluid particles, are in a constant ratio and all corresponding angles are equal.

The second condition imposed is that there be a kinematic similarity, *i.e.*, that the velocities and accelerations have a definite and constant ratio at all corresponding points. If the time required to traverse the distance L_1 is t_1 and that for L_2 is t_2, the choice of a time ratio fixes the ratios of velocities and accelerations. The kinematic relationships are

$$
\begin{aligned}
t_1 b_t &= t_2 \\
V_1 b_v &= V_2 = \frac{L_2}{t_2} = b_v \frac{L_1}{t_1} \\
b_v &= b_l b_t^{-1} = \text{ratio of velocities} \\
a_1 b_a &= a_2 = \frac{L_1}{t_1^2} b_a = \frac{L_2}{t_2^2} \\
b_a &= b_l b_t^{-2} = \text{ratio of accelerations}
\end{aligned}
$$

Here a and V are the acceleration and velocity and b indicates the ratio, at similar points, of the quantity shown as a subscript. The quantities L and t may be differential or finite.

In order to maintain motion in accordance with these kinematic relationships, certain forces are necessary, and the ratio of these forces at corresponding points in the two systems is fixed. Considering first the inertial reactions which would correspond to the forces F_{n1} and F_{n2} in Fig. 50,

$$\frac{F_{n2}}{F_{n1}} = \frac{m_2 \dfrac{V_2^2}{r_2}}{m_1 \dfrac{V_1^2}{r_1}} = b_m b_l b_t^{-2}$$

But the masses of corresponding volumes are in the ratio

$$b_m = \frac{m_2}{m_1} = \frac{\rho_2}{\rho_1} \frac{(\text{volume})_2}{(\text{volume})_1} = b_\rho b_l^3$$

and therefore

$$b_f = b_\rho b_l^4 b_t^{-2} \tag{4.9}$$

The viscous forces, which would act tangentially and may be considered as corresponding to the forces F_{t1} and F_{t2}, are in the ratio

$$\frac{F_{t2}}{F_{t1}} = \frac{\mu_2}{\mu_1} \frac{(\text{area})_2 \left(\dfrac{dV}{dr}\right)_2}{(\text{area})_1 \left(\dfrac{dV}{dr}\right)_1} = b_\mu b_l^2 b_t^{-1} \tag{4.10}$$

The two points considered are located in accordance with the requirement of geometrical similarity and the velocities are in a definite ratio. Hence, for geometrical similarity of the remainder

of the path, it is necessary that the resultant of the forces act in the same direction, *i.e.*, $\theta_1 = \theta_2$. Imposing this restriction,

$$\tan \theta_1 = \frac{F_{n1}}{F_{t1}} = \tan \theta_2 = \frac{F_{n2}}{F_{t2}}$$

Rearranging these equations,

$$\frac{F_{t2}}{F_{t1}} = \frac{F_{n2}}{F_{n1}}$$

and therefore from Eqs. (4.9) and (4.10),

$$b_\rho b_l{}^4 b_t{}^{-2} = b_\mu b_l{}^2 b_t{}^{-1}$$

This equation states the conditions for dynamical similarity of flow. Rearranging,

$$\frac{b_l b_l b_t{}^{-1} b_\rho}{b_\mu} = \frac{b_l b_V b_\rho}{b_\mu} = 1$$

Substitution of the quantities for the two systems in place of the ratios gives

$$\frac{L_1 V_1 \rho_1}{\mu_1} = \frac{L_2 V_2 \rho_2}{\mu_2} \tag{4.11}$$

Equation (4.11), which is the equation of Reynolds, states that similarity of motion should exist in two flow systems whenever the velocities, densities, and viscosities at all similarly located points are so related as to give equal values of this group of variables.

The necessity for assuming geometrical and kinematical similarity is now apparent. The derivation was based upon the motion of certain masses of fluid such as the units partaking in the turbulent exchange characteristic of turbulent flow. Ordinarily it is not possible to follow these motions but this is unnecessary since all distances are in the same ratio and one is free to use as the characteristic length the diameter of a pipe, the chord of an airfoil, or any length which may be known. Similarly, any convenient velocity may be chosen so long as the corresponding velocity is used for both systems.

Several features of this criterion of similarity should be noted. In the first place, it was assumed that only viscous forces and inertia were involved. If other types of forces, such as the

gravitational force of the earth, have appreciable effects, their magnitude at all points in the two systems must be in the ratio given by Eqs. (4.9) and (4.10). Fortunately, for flow in closed tubes and around objects deeply submerged in fluid, gravity and surface tension are not important and Eq. (4.11) is found experimentally to be valid for isothermal conditions. However, in flows with appreciable changes in temperature and density, the force of gravity may cause convection currents and an additional criterion is necessary. As regards the details of the flow pattern, equality of Reynolds number indicates that the fluctuating components of velocity, u' and w', and the average component $\bar{u}$ at corresponding points in two systems will be in the same relationship to the characteristic velocity, and the thickness of the laminar layer will be the same percentage of the characteristic length. In other words, the flow pattern will be similar in all details.

An examination of the derivation given above reveals the fact that the Reynolds number is proportional to the ratio of inertia and viscous forces. A high value of Reynolds number indicates inertia effects large in comparison with viscous forces and vice versa. Equality of Reynolds number in two geometrically similar flow systems implies equality of this ratio, not only in the main flow but in the details of the secondary flows.

47. Dimensional Analysis.—The term *dimensional analysis* is often used as though it were identical with *dynamical similarity* although they are basically different concepts. Dimensional analysis is a mathematical tool which may be used to determine the general form of physical equations, whether dynamical or not, and hence is a means of predicting the conditions necessary for dynamical similarity provided that the variables entering the problem are known. The details of the method of dimensions have been worked out largely by Rayleigh, Bridgman, and Buckingham, and students interested in a basic knowledge of this subject are referred to their works. In the treatment given here, mathematical rigor has been sacrificed in part for the sake of simplicity.

An examination of the quantities appearing in physical equations reveals that they are of two types, which we shall call primary and secondary. The primary quantities are those

which are considered to be of irreducible simplicity and are, in mechanical problems, length, mass, and time, or force, length, and time. The measurement of the primary quantities is made by direct comparison with a standard and the result is expressed in two parts, namely a number and a unit, *e.g.*, 10 ft. The number is inversely proportional to the size of the unit and has no significance unless the unit of measurement is specified. The numbers associated with secondary quantities are not in general determined by a direct comparison with a standard but by indirect methods. A velocity is measured by observing a length and a time and dividing the numbers associated with these quantities but the result still contains the units of length and time.

It is to be noted that both secondary and primary quantities are subject to the restriction that the ratio of two quantities of the same type shall be independent of the units of measurement chosen. If a certain length is twice another length, this ratio must be true whether the measurements are expressed in feet, meters, or light-years. On the basis of this restriction, Bridgman shows that the secondary quantities are of the form

$$f = Cq_1^a q_2^b q_3^c$$

where q_1,q_2,q_3 = the primary units,
C = a constant usually taken as unity,
$a,b,$ and c = constant powers.

The exponent of the primary quantity is by definition the "dimension" of the secondary quantity in the primary quantity.

The dimensions of a secondary quantity do not define its nature as is shown by writing down the dimensions of work and moment.

$$\text{Work} = F \cdot L$$
$$\text{Moment} = F \cdot L$$

Both relations are true but it is evident that work is not identical with moment.

Dimensional analysis is based upon the proposition that *dissimilar quantities cannot be added to form true physical relations* and hence, that all the members of a physical equation have the same dimensions. This property is called *homogeneity*. As an example of a nonhomogeneous equation we have

$$\text{Apples} + \text{pears} = \text{dollars}$$

which may have some meaning but which is not the kind of equation involved in physical problems. A somewhat less evident example is obtained by adding the equations for free fall:

$$\left.\begin{aligned} V &= gt \\ L &= \tfrac{1}{2}gt^2 \end{aligned}\right\} V + L = gt + \tfrac{1}{2}gt^2$$

This equation is not dimensionally homogeneous but is true for all systems of units.

TABLE 1.—DIMENSIONS OF PHYSICAL QUANTITIES

Quantity	Dimensions			Equation
	L	M	T	
Length	1	0	0	L
Time	0	0	1	T
Mass	0	1	0	M
Velocity	1	0	−1	L/T
Acceleration	1	0	−2	L/T^2
Density	−3	1	0	M/L^3
Force	1	1	−2	ML/T^2
Unit weight	−2	1	−2	M/L^2T^2
Moment	2	1	−2	ML^2/T^2
Momentum	1	1	−1	ML/T
Viscosity	−1	1	−1	M/LT
Kinematic viscosity	2	0	−1	L^2/T
Modulus of elasticity	−1	1	−2	M/LT^2
Pressure	−1	1	−2	M/LT^2
Work	2	1	−2	ML^2/T^2
Kinetic energy	2	1	−2	ML^2/T^2
Power	2	1	−3	ML^2/T^3
Head	1	0	0	L

The method of dimensional analysis developed by Rayleigh can best be demonstrated by means of an example. Suppose that experiments are to be performed on the period of a simple pendulum and that the experimenter wishes to predict the probable form of the equation to aid in planning the experiment and in representing the data. It is assumed that this hypothetical experimenter is unfamiliar with the complete dynamical analysis. From a consideration of the problem he concludes that the period, t, depends upon the length of the arm, l, the

mass of the bob, m, and the gravitational force per unit mass, g. Steps in the analysis would then be as follows:

$$t \propto l^x g^y m^z$$

It is necessary to find a combination of the exponents x,y, and z which will give the right-hand member the net dimensions of time since this is necessary if the equation is to be homogeneous. Substituting for each quantity its dimensional formula,

$$\begin{aligned} T &= L^x L^y T^{-2y} M^z \\ &= L^{x+y} T^{-2y} M^z \end{aligned}$$

To satisfy this equation, the relationship between x, y, and z must be as follows:

$$\begin{array}{lll} \text{Time:} & 1 = -2y & x = \frac{1}{2} \\ \text{Length:} & 0 = x + y & y = -\frac{1}{2} \\ \text{Mass:} & 0 = z & \end{array}$$

Substituting these values in the original equation,

$$t \propto \sqrt{\frac{l}{g}}$$

The mass has been eliminated from this particular problem, thus giving some information of a physical nature but this result was obtained because only one of the quantities contained the dimension of mass. In general, none of the original quantities will be eliminated and an incorrect assumption leads to an equally incorrect answer. The final equation merely states the general form and one can obtain the constant, which in this case is 2π, only by a detailed analysis or from experiment.

48. Fluid Friction.—Passing on to the derivation of the Reynolds criterion for similarity of flow, it is reasonable to assume that the resistance per unit area of solid boundary will depend upon the velocity of flow, the viscosity and density of the fluid flowing, and the size and roughness of the surface. To make the problem more specific, consider the solid boundary to be the interior surface of a pipe or duct, and the size to be the diameter. The equation may then be written

$$\tau \propto \mu^x \rho^y V^z D^s k^t$$

where τ is the resistance per unit area, μ the viscosity, ρ the density, V the mean velocity, D the diameter, and k a linear dimension characterizing the roughness. The dimensional equation is

$$ML^{-1}T^{-2} = M^xL^{-x}T^{-x}M^yL^{-3y}L^zT^{-z}L^sL^t$$

Equating the sums of the exponents as before,

$$\begin{aligned} &\text{Mass:} & 1 &= x + y \\ &\text{Length:} & -1 &= -x - 3y + z + s + t \\ &\text{Time:} & -2 &= -x - z \end{aligned}$$

There are three equations and five unknown exponents so that it is only possible to solve for three of them in terms of the other two. Any choice at this point will yield a correct answer but for reasons which are apparent from the result, solve for y, z, and s in terms of x and t:

$$\begin{aligned} y &= 1 - x \\ z &= 2 - x \\ s &= -x - t \end{aligned}$$

Substituting these values in the original equation,

$$\tau \propto \mu^x \rho^{1-x} V^{2-x} D^{-x-t} k^t$$

Combining quantities having the same exponents,

$$\tau \propto \rho V^2 \left(\frac{\mu}{VD\rho}\right)^x \left(\frac{k}{D}\right)^t \tag{4.12}$$

The term ρV^2 has the same dimensions as τ and the remaining terms are arranged in dimensionless groups so that the resulting equation is homogeneous. The unknown exponents x and t can be determined only by direct analysis or experiment.

The form of Eq. (4.12) is really incorrect in that it implies that single values of x and t should suffice to represent the experimental data. A more exact derivation would lead to the form

$$\frac{\tau}{\rho V^2} = \phi\left(\frac{VD\rho}{\mu}, \frac{k}{D}\right) \tag{4.13}$$

where the unknown function ϕ should be considered as the product of two infinite series in the dimensionless groups $VD\rho/\mu$ and k/D. From the method of analysis, it is evident that the inclusion of additional linear dimensions k_1, k_2, . . . representing the shape or roughness of the boundaries would have resulted in dimensionless ratios k_1/D, k_2/D, and so forth. All members of Eq. (4.13) are dimensionless, and the right-hand member is seen to include the Reynolds number and a term representing shape. A test of the validity of the equation would consist in plotting $\tau/\rho V^2$ against Re for equal values of k/D measured during the flow of any true fluid. Considering specifically the problem of pipe resistance, one can ensure a constant value of k/D by passing water, air, oil, and other fluids through the same pipeline. Under such conditions, the measured values should all lie along the same curve regardless of the fluid used. It is specifically in this respect that the Reynolds number proves so valuable in fluid mechanics for, by use of it, all data on frictional resistance are generalized in such a way that it is now only necessary to specify the viscosity and density rather than to have separate tables of data for each fluid.

49. Resistance of Objects.—The study of aerodynamics is concerned chiefly with the forces exerted on objects by moving fluids and it is largely because of studies in this field that our knowledge of fluid mechanics has advanced so rapidly in the last few years. Lest the impression be gained that the Reynolds number is the criterion only as regards frictional resistance, the method of dimensional analysis will be applied to obtain the equation for the dynamic force exerted on a submerged object of width D, projected area A, and roughness k. The force would depend upon these quantities and upon the velocity, viscosity, and density of the fluid. The equation would then be

$$F \propto \mu^x \rho^y V^z D^s k^t A^m$$

$$MLT^{-2} = M^x L^{-x} T^{-x} M^y L^{-3y} L^z T^{-z} L^s L^t L^{2m}$$

$$\begin{array}{ll} \text{Mass:} & 1 = x + y \\ \text{Length:} & 1 = -x - 3y + z + s + t + 2m \\ \text{Time:} & -2 = -x - z \end{array}$$

There are six unknown exponents and only three equations. Again solving with a view to the desired form of the equation,

$$z = 2 - x$$
$$y = 1 - x$$
$$m = 1 - \frac{x}{2} - \frac{s}{2} - \frac{t}{2}$$
$$F \propto \rho A V^2 \left(\frac{\mu}{V\rho A^{\frac{1}{2}}}\right)^x \left(\frac{D}{A^{\frac{1}{2}}}\right)^s \left(\frac{k}{A^{\frac{1}{2}}}\right)^t.$$

The conditions for similarity are that the three dimensionless groups have the same value simultaneously so that D must be proportional to $A^{\frac{1}{2}}$, a condition for geometrical similarity, and $A^{\frac{1}{2}}$ may be replaced by D in the first and third groups. Dividing through by $\rho A V^2$ and rewriting the equation in the correct functional form,

$$\frac{F}{\rho A V^2} = \phi\left(\frac{VD\rho}{\mu}, \frac{k}{D}, \frac{D}{A^{\frac{1}{2}}}\right)$$

The left-hand term is a dimensionless force coefficient which should have identical values whenever the other three groups have the same value simultaneously, regardless of the fluid used or the absolute dimensions of the object.

50. Buckingham's Π-theorem.—The three examples of dimensional analysis may now be used to illustrate at least the elements of Buckingham's Π-theorem which is more convenient than Rayleigh's method in application. In each case cited above, the number of dimensionless groups found was equal to the number of variables minus the number of primary quantities represented in these variables. In the problem of the period of a pendulum, four variables were assumed, including among them length, mass, and time, and one dimensionless group resulted. If the mass of the bob had been omitted, the three variables included among them only length and time. Buckingham showed that this relationship between the dimensionless groups, the number of variables, and the number of primary quantities was always valid and that the dimensionless groups could be obtained as follows:

1. Choose from among the variables a number of variables equal to the number of primary quantities and including all of the primary quantities.

2. Write dimensional equations combining the variables chosen in (1) with each of the others in turn.

3. Combine the dimensionless groups in any convenient manner subject to the condition that their number equals the number of variables minus the number of primary quantities.

Applying this method to the problem of the force exerted on an object, there are seven variables, the dimensions of which include length, mass, and time. Therefore, there will be four dimensionless groups. Choosing ρ, A, and V in accordance with (1) and indicating the dimensionless groups by M, the equations are

$$\Pi_1 = \rho^{a_1} A^{b_1} V^{c_1} \mu$$
$$0 = M^{a_1} L^{-3a_1} L^{2b_1} L^{c_1} T^{-c_1} M L^{-1} T^{-1}$$
$$\text{Mass:} \quad 0 = a_1 + 1$$
$$\text{Length:} \quad 0 = -3a_1 + 2b_1 + c_1 - 1$$
$$\text{Time:} \quad 0 = -c_1 - 1$$
$$a_1 = -1$$
$$c_1 = -1$$
$$b_1 = -\tfrac{1}{2}$$
$$\Pi_1 = \frac{\mu}{\rho V A^{\frac{1}{2}}}$$

Similarly for the other dimensionless groups,

$$\Pi_2 = \rho^{a_2} A^{b_2} V^{c_2} F = \frac{F}{\rho A V^2}$$
$$\Pi_3 = \rho^{a_3} A^{b_3} V^{c_3} D = \frac{D}{A^{\frac{1}{2}}}$$
$$\Pi_4 = \rho^{a_4} A^{b_4} V^{c_4} k = \frac{k}{A^{\frac{1}{2}}}$$

In accordance with (3), $A^{\frac{1}{2}}$ may be replaced by D in Π_1 and Π_4. This modification changes the Π's only by a numerical constant. Since Π_2 is the resistance coefficient, it is convenient to write the equation in the following form:

$$\Pi_2 = f(\Pi_1, \Pi_3, \Pi_4)$$

or

$$\frac{F}{\rho A V^2} = f\left(R_e, \frac{D}{A^{\frac{1}{2}}}, \frac{k}{D}\right)$$

51. Numerical Values of Reynolds Number.—In computing the Reynolds number, any length may be chosen since the flow systems to be compared are geometrically similar, but it is usual

to choose a convenient length, such as the diameter of a pipe. Similarly, any velocity may be used, and one chooses a velocity which is generally part of the known or directly measurable data. The radius of a pipe might just as well be used but the resulting Reynolds number would differ by a factor of 2. The maximum velocity might be used in place of the mean velocity in a pipe but this is inconvenient because it is usually the mean velocity which is known. It is also convenient for comparative purposes to use the same relative dimensions for all systems of the same type. For example, in computing the Reynolds number for venturi meters and diaphragm orifices, it would be convenient for comparison if the length dimension for both were taken as the diameter of the smallest section. However, this principle is generally not followed, the Reynolds number of the orifice being based on the pipe diameter while that of the venturi meter utilizes the throat diameter. In short, the numerical value of the Reynolds number should always be accompanied by a detailed statement of the quantities used in its computation.

Ordinarily, problems in pipe flow specify the diameter and the weight or volumetric rate of discharge. Computation of the Reynolds number is facilitated by use of the computation forms:

Liquids: $$\mathrm{Re} = \frac{VD}{\nu} = \tag{4.14}$$

Gases: $$\mathrm{Re} = \; VD\rho \quad \frac{W}{Aw} \cdot \frac{D}{\mu} \cdot \frac{w}{g} \quad 4W \tag{4.15}$$

The equation for gases is especially convenient because μ is the only variable for a certain weight rate of discharge, W, in a pipe of diameter, D. The absolute viscosity, μ, depends largely on temperature and hence Re is a constant for isothermal flow of gases. The numerical constant, $4/\pi$, is retained so as to give numerical values consistent with the basic definition. The remaining terms have the following significance: V is the average velocity over a section of diameter D, Q is the volumetric rate of discharge, ν is the kinematic viscosity, and g is the weight per unit mass.

52. Experimental Verification.—The Reynolds number is supposedly the criterion of similarity so long as forces other than viscous do not play an important part and therefore Eq. (4.13) should apply to laminar flow in pipes. From Eq. (4.7),

$$\frac{\tau}{\rho V^2} = \frac{p'}{\rho V^2}\frac{\pi D^2}{4\pi D} = 8\frac{\mu}{VD\rho} = \frac{8}{\mathrm{Re}} \tag{4.16}$$

Equation (4.16) is found experimentally to apply to all isothermal viscous flows so long as the roughness is not appreciable

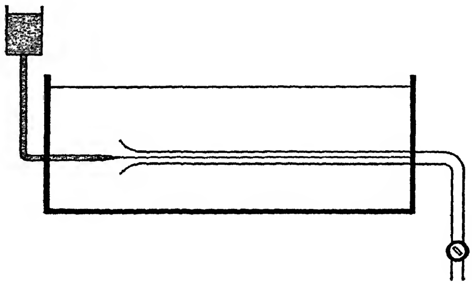

FIG. 51.

as compared with the diameter of the pipe. It has the form specified by Eq. (4.13) with the simplification that k/D has been eliminated.

The next important check on the validity of the Reynolds number as a criterion of similarity is its reliability as an indicator of the transition from laminar to turbulent flow. This verification can best be illustrated by describing Reynolds' own experiments which employed a bellmouthed glass tube leading from a reservoir containing water at rest. By introducing a stream of colored liquid at the bellmouthed inlet, the motion of the fluid particles was made visible and it was found that on increasing the velocity of flow progressively there was a definite velocity at which the stream of coloring matter broke up into eddies and filled the pipe. The velocity at which this transition takes place is known as the *critical velocity*. When the values of the velocity were plotted against the head loss, it was found that there were *two* critical velocities, one corresponding to the change from viscous to turbulent flow when the flow is gradually increased and another for the

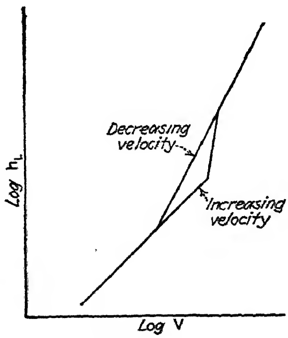

FIG. 52.

change from turbulent to viscous flow when the flow is decreased. The results are shown in Fig. 52.

As a result of his experiments Reynolds came to the following conclusions regarding the conditions tending to produce the two types of flow:

DIRECT OR LAMINAR MOTION	SINUOUS OR TURBULENT MOTION
Viscosity or fluid friction which continually destroys disturbances.	Variation of velocity across a stream as when a stream flows through still water.
A free surface.	Solid boundary walls.
Converging solid boundaries.	Diverging solid boundaries.
Curvature with the velocity greatest at the outside.	Curvature with velocity greatest on the inside.

For usual conditions, the flow is viscous for $VD/\nu < 2000$, and turbulent for $VD/\nu > 2700$, and between these values is a region of uncertainty. Viscous flow can be maintained up to much higher values of VD/ν by careful manipulation, and turbulent flow will result at Reynolds numbers much below 2000 if the inlet conditions are unfavorable. Indicating the critical velocity as V_c, the boundary between streamline and turbulent flow is approximately

$$V_c = \frac{2500 \cdot \nu}{D}$$

The fact that the transition from laminar to turbulent flow in pipes does not take place at a single value of the Reynolds number, but rather over a broad range, does not indicate that it is not the proper criterion. The variation results from lack of complete similarity as can be shown by using the same pipe for different fluids, in which case the transition occurs in a relatively narrow zone. Similar experiments on the flow around objects with the same relative roughness also substantiate the Reynolds number as the proper criterion.

53. Resistance in Pipes.—The basic equation commonly used in computing friction losses in pipes is

$$h_L = \frac{fL}{D}\frac{V^2}{2g} \text{ (Weisbach's equation)} \qquad (4.17)$$

which is applicable to both laminar and turbulent flow if the

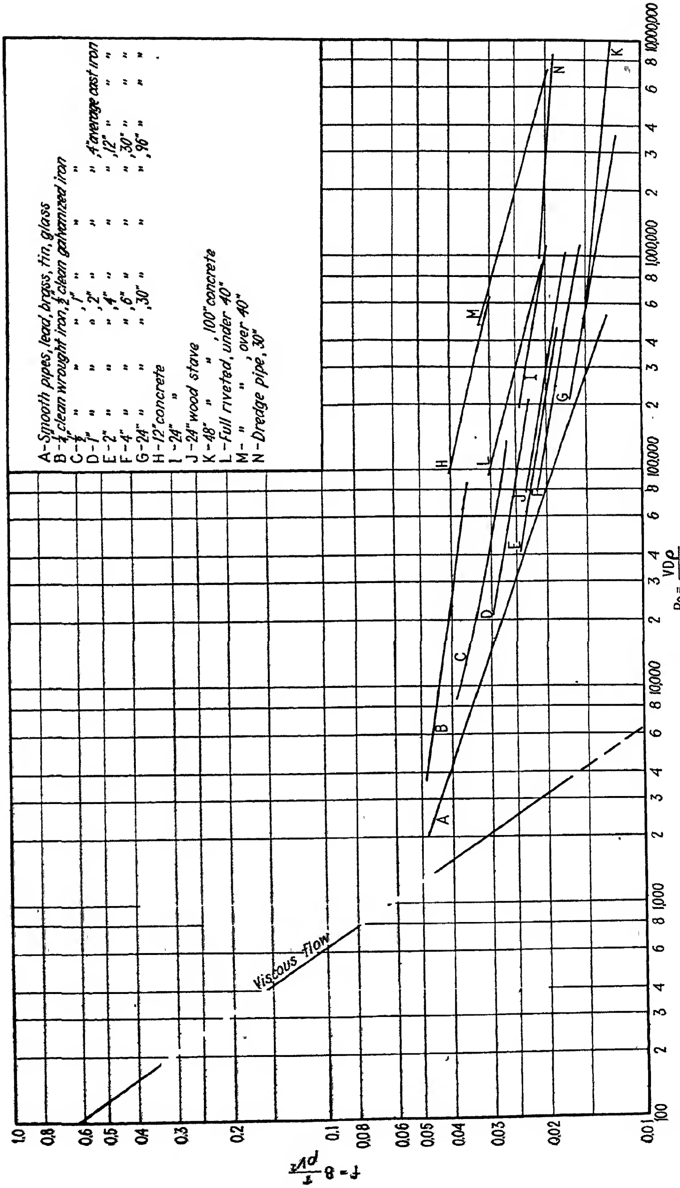

FIG. 53.—Relation of friction factor and Reynolds number for circular pipes of different diameters and roughnesses.

coefficient f is properly chosen. Solving for f and inserting the tractive force at the wall,

$$\tau = \frac{\pi R^2 w h_L}{2\pi R L}, \qquad f = 8 \cdot \frac{\tau}{\rho V^2}$$

The friction factor is numerically eight times greater than $\tau/\rho V^2$ but is of the same nature and must depend upon the same variables. Consequently

$$f = \phi\left(\text{Re}, \frac{k}{D}\right)$$

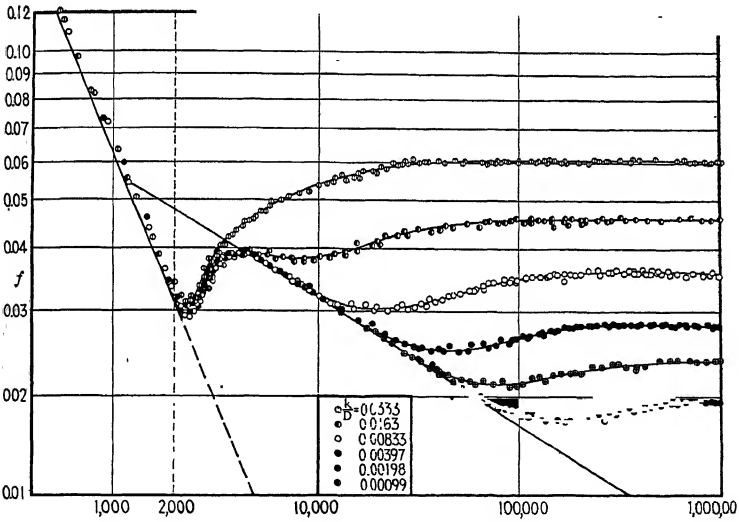

FIG. 54.—Nikuradse's experiments, showing the effect of roughness on the transition from laminar to turbulent flow.

Figure 53, showing the course of this function in the laminar and turbulent regions, substantiates the validity of the Reynolds number as the criterion of similarity.

Experiments by Nikuradse at Göttingen (Fig. 54) show the effect of roughness on both the transition from laminar to turbulent flow and on the turbulent friction coefficient. In the laminar region, the roughness has only a negligible effect and Poiseuille's equation is valid. The transition from laminar to

turbulent flow occurs at the same Reynolds number (about Re = 2500) regardless of the roughness but in the turbulent region the roughness has a marked effect. Not only is the friction coefficient greater for increased roughness but the shape of the curves is modified. Each of the curves shown ultimately attains a constant value of f independent of Reynolds number, and therefore of viscosity, and the Reynolds number at which f becomes constant decreases with increasing roughness. When f is constant, the turbulence is said to be *fully developed.*

Generalization of the phenomenon of fluid friction by means of the Reynolds number does not greatly increase our ability to predict friction losses for the flow of water, but it does make possible the application to other fluids of the great mass of experimental data obtained with water. A number of summaries of the available data have been published and a comprehensive review will not be attempted here, but it is important to show the relationship between the purely empirical formulas and the Reynolds equation. Taking the equation of Blasius as an example, the general form is

$$f = \frac{K}{\mathrm{Re}^{\frac{1}{4}}} = \frac{K\nu^{0.25}}{V^{0.25}D^{0.25}}$$

Substituting this value of f in Eq. (4.17),

$$h_L = \frac{KL\nu^{0.25}}{2g} \cdot \frac{V^{1.75}}{D^{1.25}}$$

Equations of this type are often found in the literature on hydraulics. The form results from the fact that over limited ranges of Reynolds number, the friction factor curve may be represented as a straight line on a logarithmic diagram. They are interpolation formulas only and should not be applied for Reynolds numbers outside the range of the original experiments from which they were derived. Many similar formulas for friction losses in pipes and channels have been suggested. A number of these are included in Chap. VII.

Another substantiation of the Reynolds number as the criterion of similarity is the data of Stanton and Pannell on the ratio of the mean to the maximum velocity (Fig. 55). This ratio is a measure of the similarity of the velocity distribution,

and the fact that it is the same at equal values of the Reynolds number with different fluids in pipes of different diameter

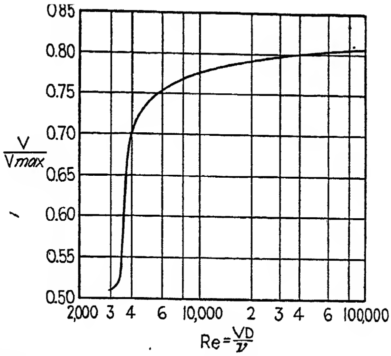

Fig. 55.—Effect of Reynolds number on the ratio of mean to maximum velocity in a pipeline.

indicates similarity, not only as regards the over-all friction factor, but the details of the secondary flows as well.

54. Hydraulic Radius.—For noncircular pipes, the friction losses may be computed approximately from data on circular

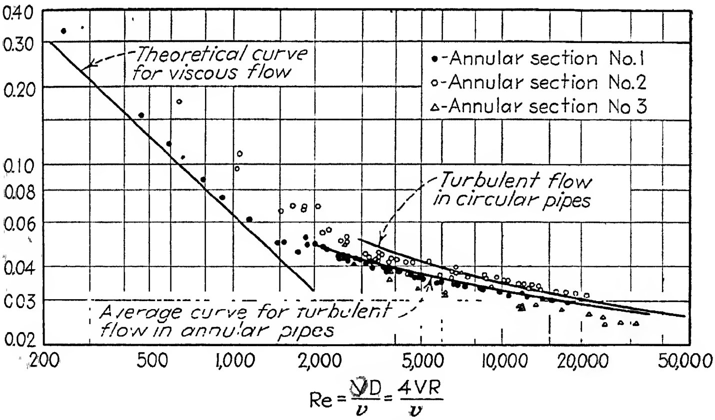

Fig. 56.—Values of f for circular and annular pipes. Hydraulic radius of annular pipes: No. 1, .00833 ft.; No. 2, .0160 ft.; No. 3, .02085 ft.; of circular pipes, .0288 ft. and .0431 ft.

pipes by use of the hydraulic radius, which is defined as the ratio of the flow area to the wetted perimeter. The hydraulic

radius of a circular pipe is $D/4$ and therefore the diameter of an equivalent circular pipe is $4R$. The pertinent equations are

$$\mathrm{Re} = \frac{4RV}{\nu}$$

$$h_L = \frac{fL}{4R}\frac{V^2}{2g}$$

The value of f is obtained from curves or tables for circular pipes at the computed value of Re. This method yields a fair approximation to the friction loss and is more accurate in turbulent than laminar flow. Figure 56 shows the results of experiments by Kratz, Macintire, and Gould.

55. Isothermal Steady Flow of Liquids in Pipes.—The term *pipe* is used to indicate a flow passage of constant area and shape. The known quantities are Q, the rate of discharge; D, the diameter; ν, the kinematic viscosity; and the nature of the pipe surface. The method of applying the Reynolds number to the computation of friction losses can best be illustrated by examples.

1. Compute the pressure drop required to pump 18,000 lb. per hr. of castor oil at 60 deg. Fahr. through a horizontal 4-in. cast-iron pipe 1000 ft. long.

From physical properties of castor oil (Appendix I),

$$\nu_{60^\circ} = 1.65 \times 10^{-2} \text{ ft.}^2 \text{ per sec.}$$

$$w = 60.17 \text{ lb. per cu. ft.}$$

$$\mathrm{Re} = \frac{4 \times 0.0831}{3.142 \times 0.333 \times (1.65 \times 10^{-2})} = 19.3$$

The flow is laminar and, from Eq. 4.16, the value of $f/8$ or $\tau/\rho V^2$ is 0.415.

$$h_L = \frac{8 \times 0.415 \times 1000 \times (0.955)^2}{64.4 \times 0.333} = 141 \text{ ft. of castor oil}$$

$$\Delta p = hw_{\text{oil}} = \frac{141 \times 60.17}{144} = 59.0 \text{ lb. per sq. in.}$$

If the pipe in this problem were not horizontal, h_L would be unchanged but the pressure drop would be modified by the change in elevation according to the equation

$$h_L = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)$$

2. Compute the pressure drop necessary to discharge 8000 lb. of glycerin per hour at 68 deg. Fahr. through 1050 ft. of $1\frac{1}{2}$-in. cast-iron pipe.

$$\text{Sp. gr.} = 1.26, \qquad \nu = 0. \qquad {}^2 \text{ per sec.}$$

$$\frac{8000}{3600 \times 62.4 \times 1.26} = 0.0282 \text{ cu. ft. per sec.}$$

$$\frac{Q}{A} = 2.3 \text{ ft. per sec.}$$

$$\text{Re} = \frac{Vd}{\nu} \qquad \frac{2.3 \times 1.5}{12 \times 0.00861} \qquad 33.4$$

From Fig. 53,

$$\frac{\tau}{\rho V^2} \qquad h_L \quad \frac{\qquad}{2gD}$$

$$h_L \quad \frac{8 \times 0.24 \times 1050 \times 2.3^2 \times 12}{64.4 \times 1.5} = 1325 \text{ ft. o}$$

$$\Delta p \quad \frac{1325 \times 62.4 \times 1.26}{144} = 723 \text{ lb. per sq. in.}$$

3. Compute the pressure drop necessary to discharge 2.5 cu. ft. per sec. of water at 70 deg. Fahr. through a 12-in. cast-iron pipe 4000 ft. long.

$$\nu_{70^\circ} = 1.07 \times 10^{-5},$$
$$w = 62.3 \text{ lb. per cu. ft.}$$
$$\text{Re} = \frac{4Q}{\pi D\nu} = \frac{4 \times 2.5}{3.14 \times 1 \times 1.07 \times 10^{-5}} \qquad 297{,}000.$$

From the Reynolds curve for 12-in. cast-iron pipe (Fig. 53),

$$f = 0.019, \qquad V = \frac{Q}{A} = 3.18 \text{ ft. per sec.}$$

$$_L \quad \frac{0.019 \times 4000 \times 3.18^2}{64.4 \times 1} \qquad 12.0 \text{ ft. of water}$$

$$\Delta p \quad \frac{12.0 \times 62.3}{144} = 5.2 \text{ lb. per sq. in.}$$

56. Flow of Gases in Horizontal Pipes.—The drop in pressure necessary to overcome the frictional resistance is accompanied by an expansion of the gas and a resulting increase in velocity in the direction of flow. The pressure drop must be sufficient to balance the resisting shear at the walls and to accelerate the flow. From the momentum relations previously considered, the basic equation is

$$A\,dp = -\left(\pi D\tau\,dx + \frac{W}{g}dV\right) \qquad (4.18)$$

where A is the area of the pipe, D is the diameter of the pipe, τ is the resistance per unit surface area, and W is the weight per unit time (see Fig. 57).

The value of τ will vary from point to point along the pipe but at each point it will be approximately

$$\tau = \frac{f}{8}\rho V^2$$

where f is the friction coefficient obtained from the Reynolds curve (Fig. 53). The Reynolds number is

$$\text{Re} = \frac{4W}{\pi g \mu D}$$

Substituting for τ in Eq. (4.18) and simplifying,

$$dp = -\frac{fwV^2}{2gD}dx - \frac{W}{Ag}dV \tag{4.19}$$

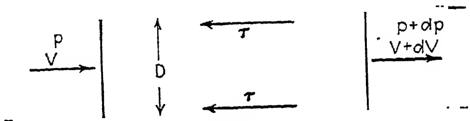

FIG. 57.

Integration of Eq. (4.19) requires a knowledge of the type of expansion involved. In long, uninsulated pipes the flow may approach isothermal conditions, while good insulation will cause it to be very nearly adiabatic. In some cases, heat may be transferred into the fluid from the surroundings, and so for the general equation one must assume a polytropic expansion, $pv^n = \text{const.}$, with the value of n depending on the process. Steps in deriving the final flow equation on the basis of Eq. (4.19) are the following:

$$\left.\begin{aligned} w &= \frac{1}{v} = \frac{1}{v_1}\left(\frac{p}{p_1}\right)^{\frac{1}{n}} \\ V &= \frac{W}{wA} = \frac{Wv_1}{A\left(\frac{p}{p_1}\right)^{\frac{1}{n}}} \\ dV &= -\left(\frac{1}{n}\right)\frac{Wv_1p_1^{\frac{1}{n}}}{Ap^{\frac{n+1}{n}}}dp \end{aligned}\right\} \tag{4.20}$$

Substituting in Eq. (4.19),

$$dp = -\frac{f}{2gD}\frac{W^2}{A^2}v_1\left(\frac{p_1}{p}\right)^{\frac{1}{n}}dx + \frac{W^2 v_1 p_1^{\frac{1}{n}}}{nA^2 g p^{\frac{n+1}{n}}}dp$$

Multiplying by $p^{\frac{1}{n}}$ and integrating on the assumption that f is constant,

$$\int_{p_1}^{p_2} p^{\frac{1}{n}}\,dp = -\frac{f v_1 W^2}{2g\,DA^2}p_1^{\frac{1}{n}}\int_0^L dx + \frac{W^2 v_1 p_1^{\frac{1}{n}}}{nA^2 g}\int_{p_1}^{p_2}\frac{dp}{p}$$

$$\frac{n}{n+1}\left(p_2^{\frac{n+1}{n}} - p_1^{\frac{n+1}{n}}\right) = -f\frac{L}{D}\frac{W^2 v_1 p_1^{\frac{1}{n}}}{2A^2 g} + \frac{W^2 v_1 p_1^{\frac{1}{n}}}{nA^2 g}\ln\frac{p_2}{p_1} \quad (4.21)$$

Dividing through by $-p_1^{\frac{1}{n}}$

$$\frac{n}{n+1}\left[p_1 - p_2\left(\frac{p_2}{p_1}\right)^{\frac{1}{n}}\right] = \frac{W^2 v_1}{gA^2}\left(\frac{fL}{2D} + \frac{1}{n}\ln\frac{p_1}{p_2}\right) \quad (4.22)$$

The pressure p_2 is obtained by successive approximations, assuming first that $\ln p_1/p_2 = 0$. The computation form is

$$p_2 = p_1\frac{v_2}{v_1} - \frac{n+1}{n}\cdot\frac{W^2 v_2}{gA^2}\left(\frac{fL}{2D} + \frac{1}{n}\ln\frac{p_1}{p_2}\right)$$

The pressures are absolute, the logarithm is to the base e, and the value of n depends upon the process.

The equation for the isothermal flow of gases may be obtained from Eq. (4.21) by inserting $n = 1$ and using the gas law, $pv = RT$:

$$\frac{1}{2}(p_1^2 - p_2^2) = \frac{W^2 RT}{gA^2}\left(\frac{fL}{2D} + \ln\frac{p_1}{p_2}\right) \quad (4.23)$$

From Eq. (4.15) the Reynolds number will be the same at all points along the pipe since μ depends only on temperature, which in this case is constant.

The fact that the Reynolds number varies with temperature in accordance with Eq. (4.15) shows that Eq. (4.22) is strictly applicable only to isothermal flow. However, pressure gradients are appreciable in gases only in the turbulent region where the

friction factor is relatively constant and Eq. (4.22) should be valid for nearly all problems.

The equation for gas flow can be simplified somewhat when the pressure drop is comparatively small. Under this condition, $\ln \frac{p_1}{p_2}$ is very small, and may be neglected. Writing $p_2 - p_1 = \Delta p$, Eq. (4.23) becomes

$$\frac{p_1^2 - (p_1 - \Delta p)^2}{2} = \frac{W^2RT}{A^2 2g} f \frac{L}{D}$$

Expanding and neglecting $\overline{\Delta p}^2$,

$$p_1 \Delta p = \frac{W^2RT}{2A^2g} f \frac{L}{D}$$

or since

$$\frac{RT}{p_1} = v_1 = \frac{1}{w_1} \qquad \text{and} \qquad \frac{W^2}{A^2w^2} = V^2$$

$$\frac{\Delta p}{w} = h_L = \frac{fL}{D}\frac{V^2}{2g}$$

which is identical with Eq. (4.17) for liquids. In other words, when the pressure drop is small as compared with the absolute pressure, the flow may be treated as incompressible. The error introduced may be computed from Eq. (4.22) in case there is uncertainty as to whether the pressure drop is negligible.

57. Flow of Gas in Inclined Pipes.—In dealing with the flow of gases, the inclination of the pipe and the direction of flow may be important in determining the pressure changes. With downward flow, the drop in pressure due to friction losses is opposed to the increase of pressure due to the elevation, while for upward flow both the friction drop and the change in elevation tend to reduce the pressure and increase the velocity.

Referring to Fig. 58, the differential equation for the flow of a gas in the direction shown is

$$dp = -\left(w\,dz + \frac{W}{gA}dV + \frac{fwV^2}{2gD}dl\right) \tag{4.24}$$

For flow in the opposite direction (downward), Eq. (4.24) applies if dz is taken as negative and dl is positive. Equation

(4.24) can be integrated by inserting the relationships of Eq. (4.20). To obtain numerical values for the pressure drop at a certain weight rate of discharge, the exponent n must be specified and this requires successive approximations which can best be carried out by first applying Eqs. (4.17) and (4.22) in order. From the pressure difference given by Eq. (4.22) one can estimate the approximate percentage effect of the difference in elevation and determine whether Eq. (4.24) must be employed. The latter equation is seldom necessary and its integration will not be carried through here.

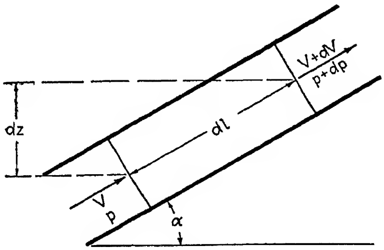

FIG. 58.

58. Experimental Results.—The equations for the flow of gases have been considered in some detail in this chapter because they define the friction factor for compressible flow and are necessary for the interpretation of experimental results. If Eq. (4.17), which is valid for liquids, is applied to experimental measurements made with gases, the friction factor f so determined is in error by an amount depending on the expansion ratio. The pressure gradient results from the combined effect of friction and acceleration of the flow, but use of Eq. (4.17) is equivalent to the assumption that friction alone is operative, resulting in friction coefficients which are too large. A proper separation of the effect of compressibility, such as that represented in Eq. (4.23), should give friction coefficients which are independent of the pressure and which are identical with values measured for liquids at the same Reynolds number and relative roughness. It was assumed in deriving these equations that the rate of expansion of the gas did not modify the value of f. The close agreement between measured values using superheated steam, air, oil, water, and other fluids shows that this assumption is justified in the usual range of conditions. However, the rate

of expansion may affect the momentum interchange and thus modify the friction coefficient at velocities approaching acoustic values. There are some experiments which indicate such a difference and this problem is one worthy of careful investigation.

59. Turbulent Flow.—The quantitative theory of turbulent flow, developed largely by Osborne Reynolds, G. I. Taylor, L. Prandtl, and Th. von Kármán, is being modified and refined so rapidly that a detailed summary is unwarranted here. However, this theory has such practical significance that engineers should be familiar with at least its basic concepts. The treatment given here is greatly simplified and the reader is referred to the many recently published summaries for more complete details.

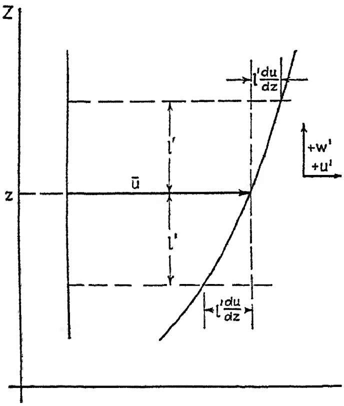

FIG. 59.

Osborne Reynolds first proposed the theory of turbulent flow which is now generally accepted. He conceived the flow at a point to be composed of an average velocity upon which are superimposed fluctuating components. To simplify the problem, consider a flow in which the velocity, averaged over a time, long as compared with the period of a single fluctuation, is parallel to the X-axis. Indicating the average velocity in this direction by $\bar{u}$ (Fig. 59), the instantaneous velocity components in two-dimensional flow are

$$u = \bar{u} + u'$$
$$w = w'$$

Here the time averages of u' and w' are zero. The instantaneous rate of transfer of the X-component of momentum per unit area at any point in the Z-direction is

$$-\rho w'(\bar{u} + u')$$

The situation is similar to that assumed in deriving the equation for the viscosity of a gas from the kinetic theory (see page 86).

The time average of this momentum transfer must correspond to a tractive force (shearing force per unit area) and the equation is

$$\tau = -\frac{}{} \quad \text{[illegible]} \quad u')dt \tag{4.25}$$

where the time interval t includes many fluctuations. Breaking this integral up into two parts, the time average of the first term is

$$\frac{1}{t}\int_0 \rho \bar{u} w' \, dt = 0$$

because $\bar{u}$ is a constant and the time average of w' is zero. Integrating the second term on the assumption that ρ is constant, Eq. (4.25) becomes

$$= -\frac{\rho}{t}\int_0^t u'w' \, dt = -\rho\overline{u'w'} \tag{4.26}$$

The expression $\overline{u'w'}$ represents the time average of the product of the instantaneous fluctuating velocity components and is not necessarily zero even though the time averages of u' and w' are zero when taken separately.

It is now necessary to investigate the degree of correlation between u' and w'. Referring to Fig. 59, if w' is positive, particles move upward from regions of lower velocity and, if they originally have an X-component of velocity equal to $\bar{u}$ at their origin, the observed value of u' when they reach their destination will be negative. Conversely, if w' is negative (downward), particles move downward from regions of higher velocity and at their destination the observed value of u' will be positive. It appears then that the product $u'w'$ is always negative and hence τ as defined by Eq. (4.26) is positive. Since two equal and opposite forces act on the fluid above and below the plane through (x,z) the positive sign merely indicates that the force tending to accelerate the fluid below this plane has been considered.

The next step in the development of the theory is to relate u' and w' to the velocity gradient. Prandtl assumed that macroscopic units of fluid move in a manner similar to the motion of molecules in a gas and that they travel through a distance perpendicular to $\bar{u}$, which corresponds to the "mean free path"

of a gas molecule. In the theory of turbulence this distance is known as the *mixing length*. On the basis of these assumptions the average absolute value of u' is $\overline{|u'|} \propto l' \frac{d\bar{u}}{dz}$. In writing this equation, the velocity gradient is assumed to be linear over the distance l'. The absolute value can be taken because it has been shown previously that the correlation between u' and w' is such that the product is always negative.

Prandtl assumed that $\overline{|w'|}$ is proportional to $\overline{|u'|}$. The reasonableness of this assumption may be demonstrated as follows: Assume that two fluid masses have arrived at the elevation z from above and below that level. The mass from above will initially have a u-component of velocity greater than the average while that from below will have a lesser velocity. The two masses will then tend to move together or draw apart and the fluid between them will either be forced out of the plane or additional fluid will be drawn in. Hence, the vertical velocities induced will be proportional to the difference in the velocity of the two masses and therefore to u'. The expression for the tractive force involves $\overline{u'w'}$ which differs from the product of the average of the absolute values of u' and w' by a constant factor. The equation for the apparent tractive force is then

$$\tau = -\text{const. } \rho\overline{|u'|} \cdot \overline{|w'|} = -\text{const. } \rho\overline{l'^2}\left(\frac{d\bar{u}}{dz}\right)^2$$

Since l' is a quantity not easily susceptible of direct measurement, the constant might as well be incorporated in a new length l. Furthermore, the sign of τ should vary with the sign of du/dz. The final equation becomes

$$\tau = \rho l^2 \left|\frac{d\bar{u}}{dz}\right| \cdot \frac{d\bar{u}}{dz} \tag{4.27}$$

If the theory is correct, the length l should be proportional to the average transverse distance moved through by the fluid masses.

The magnitude of the coefficient l may be obtained from experiments on flow in pipes, if the velocity distribution is symmetrical about the axis. Under such conditions, the tractive force is

$$\tau = \frac{r}{2}\frac{dp}{dx}$$

Inserting this value for τ in Eq. (4.27) and solving for l,

$$|l| = \frac{\sqrt{\frac{r}{2\rho}\frac{dp}{dx}}}{\left|\frac{du}{dz}\right|}$$

Experiments of Nikuradse and others show that l is given by the equation

$$\frac{l}{r} = 0.14 - 0.08\left(1 - \frac{z}{r}\right)^2 - 0.06\left(1 - \frac{z}{r}\right)$$

where z is the distance from the wall and r the radius of the tube. Expanding l,

$$l = 0.4z - 0.44\frac{z^2}{r} + \qquad (4.28)$$

von Kármán has attacked the problem of the structure of turbulent flow from the standpoint of similarity of the local flow pattern. He assumed that

a. The mechanism of turbulent interchange is independent of the viscosity except in the immediate neighborhood of the walls.

b. The principle of similarity can be applied to the mechanism of turbulent interchange in a sense analogous to that mentioned above (molecular friction), *i.e.*, the local flow pattern is statistically similar in the neighborhood of every point and only the time and length scale are variable.

On the basis of these assumptions it is shown that the velocity components u' and w' are equal to a characteristic length, l, times $d\bar{u}/dz$. The shearing stress is $\rho l^2\left(\frac{du}{dz}\right)$ and the characteristic length l is given by the equation

$$l = \frac{d\bar{u}}{\frac{d^2u}{dz^2}} \qquad (4.29)$$

Here, κ is a pure number which is a "universal constant characteristic of the turbulent interchange." In the immediate vicinity of the wall, but outside of the laminar sublayer, Eq. (4.29) reduces to

$$l = \kappa z \qquad (4.30)$$

Comparison with Eq. (4.28) shows that the value of the constant κ is about 0.4. A number of other experiments have yielded very nearly the same result.

In laminar flow, the shearing force is $\tau = \mu \frac{du}{dz}$. Stanton adopted a similar equation for turbulent flow,

$$\tau = \epsilon \frac{d\bar{u}}{dz} \tag{4.31}$$

in which he called the quantity ϵ the mechanical viscosity. It is also referred to as the transfer or *austausch* coefficient. Comparing Eqs. (4.27) and (4.31),

$$\epsilon = \rho l^2 \frac{d\bar{u}}{dz} = \rho l w' \tag{4.32}$$

which is identical in form with Eq. (4.4) but is basically different since both w' and l are dependent upon the flow conditions rather than the properties of the fluid. The magnitude of the mechanical viscosity is much greater than the molecular viscosity.

60. Laminar Sublayer.—It has been stated previously that fluids do not slip relative to a solid boundary even in turbulent flow. A layer which is in laminar flow is always attached to the boundary and the thickness of this layer may be approximated as follows: The tractive force will be nearly constant if the layer is thin and therefore the governing equation is $\tau_0 = \mu \frac{u_0}{z_0}$ where u_0 is the velocity at the edge of the laminar layer of thickness z_0, and τ_0 is the tractive force at the wall. The magnitude of the tractive force depends upon the velocity and the friction coefficient in accordance with the equation

$$\tau_0 = C_f \tfrac{1}{2} \rho u_m^2$$

Replacing u_0 by Nu_m, inserting these values and solving for z_0,

Taking $N = 0.5$ and $C_f = 0.004$ as approximate values

$$z_0 = 11 \frac{\nu}{\sqrt{\frac{\tau_0}{\rho}}} \tag{4.33}$$

von Kármán states that a reasonable limit for the distance to which the viscosity is effective is given by $C_0 = 30$ but the thickness of the layer in laminar flow appears to be given by $C_0 = 11$ (approx.) (see page 128). For a 12-in. cast-iron pipe carrying water at normal temperature and at a velocity of 10 ft. per sec., $\tau_0 = 0.6$ lb. per sq. ft., and with $C_0 = 30$, $z_0 = 5.2 \times 10^{-4}$ ft. The laminar layer is seen to be extremely thin but it nevertheless plays an important part in the phenomenon of turbulence and is a basic factor in certain related phenomena such as heat transfer.

It is now possible to give a quantitative definition of the difference between smooth and rough pipes.

The hydraulic definition of smooth pipes given by von Kármán is

$$\left(\frac{\tau_0}{\rho}\right)^{\frac{1}{2}}\frac{k}{\nu} < 3 \tag{4.34}$$

Fully developed turbulence occurs when

$$\left(\frac{\tau_0}{\rho}\right)^{\frac{1}{2}}\frac{k}{\nu} > 60 \tag{4.35}$$

and this may be taken as the quantitative definition of rough pipes. It is evident that the roughness of a certain pipe according to this definition increases with velocity and therefore with Reynolds number, and that any pipe will be hydraulically rough at a sufficiently high Reynolds number. Here, k is the diameter of the granular roughness. From Eq. (4.33) it is evident that these values correspond to

$$\frac{k}{z_0} \cong 0.27, \qquad \text{and} \qquad \frac{k}{z_0} > 5.5, \text{ respectively.}$$

z_0 is the thickness of the laminar layer computed by means of Eq. (4.33) for a certain tractive force. The significance of the term *laminar layer* when k is several times z_0 is problematical, but the two limits first expressed do have definite significance and it is perhaps best not to think in terms of the ratio k/z_0.

The quantity $(\tau_0/\rho)^{\frac{1}{2}}$ appears in a number of the equations and it is convenient to introduce at this time two generalized parameters which include this quantity. The dimensions of $(\tau_0/\rho)^{\frac{1}{2}}$ are those of a velocity and for this reason it is referred to

as the *friction velocity* and will be designated as V^*. A dimensionless group can therefore be built up as

$$U^*$$

Another dimensionless group, which is really a localized Reynolds number, is

$$\text{Re}^* =$$

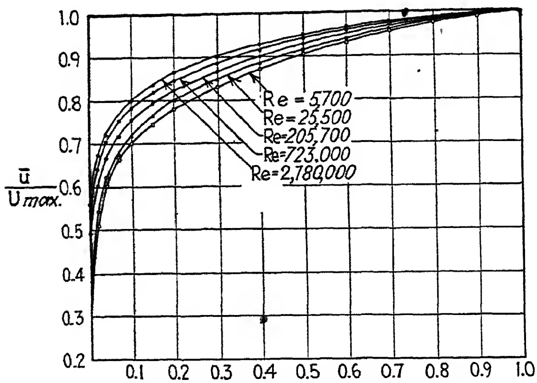

FIG. 60.—Effect of Reynolds number on velocity distribution in a circular pipe as determined by Nikuradse.

In terms of these quantities, the velocity distribution in the laminar sublayer is represented by

$$U^* = \tag{4.36}$$

61. Velocity Distribution in Pipes.—By means of the generalized parameters U^* and Re^* it is possible to construct generalized curves of the velocity distribution. Figure 60 shows the velocity distribution in *smooth*, circular pipes as measured by Nikuradse for $5700 < \text{Re} < 2{,}780{,}000$. The form changes progressively towards a U-shaped distribution as the Reynolds

increases. These velocity distributions may be reduced to a single curve as shown in Fig. 61 by plotting

$$U^* = \frac{u}{V^*} = f(\mathrm{Re}^*) = f\left(\frac{\sqrt{\frac{\tau_0}{\rho}} \cdot z}{\nu}\right) \tag{4.37}$$

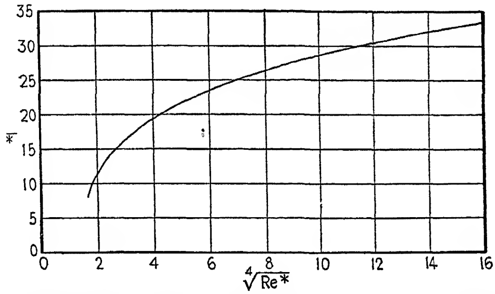

FIG. 61.—Generalized velocity curve for smooth pipes. (*Nikuradse.*)

Here, u is the velocity at distance z from the boundary. A useful relationship found by Nikuradse for smooth pipes is

$$U_{\mathrm{mean}} = U_{\mathrm{max}} - 4.08V^*$$

Equation (4.37) is restricted to smooth pipes because it is known that roughness of the boundaries affects the absolute velocity. On the basis of von Kármán's theory of the local similarity of the flow pattern in the turbulent core of a pipe, it will be shown later that the shape of the velocity curve depends on the tractive force only. Eliminating absolute velocities by considering the difference between the velocity at the center and that at point z (velocity defect), the velocity distribution in both rough and smooth pipes may be generalized by plotting

$$\frac{U_{\mathrm{max}} - u}{V^*} = \phi\left(\frac{z}{r}\right) \tag{4.38}$$

Figure 62 shows a generalized velocity deficiency curve as drawn by von Kármán from data of Donsch and Nikuradse. The theoretical equation of the curve shown in this figure will be developed later.

Equation (4.38) does not give the absolute magnitude of the velocity u as a function of z but does apply regardless of

the magnitude of the roughness. Nikuradse has proposed the generalized representation shown in Fig. 63 in which the quantity M is shown as a function of a Reynolds number which includes the roughness as a linear dimension. The quantities plotted are $\frac{V^*k}{\nu}$ and

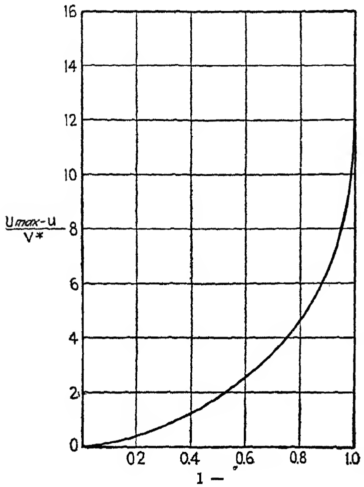

FIG. 62.—Generalized velocity deficiency curve as drawn by von Kármán from data by Donsch and Nikuradse.

$$M = \frac{u}{V^*} - 5.75 \log_{10} \frac{z}{k},$$

The data on which this graph is based are those shown in Fig. 54. For fully developed turbulence, corresponding to a constant coefficient of resistance,

$M = 8.5$ and

$$u = \sqrt{\frac{\tau_0}{\rho}}\left(5.75 \log_{10} \frac{z}{k} + 8.5\right)$$

von Kármán states that experiments of Kempf on flat plates, prepared in a manner similar to the pipes of Nikuradse, are not in agreement with this generalized roughness function.

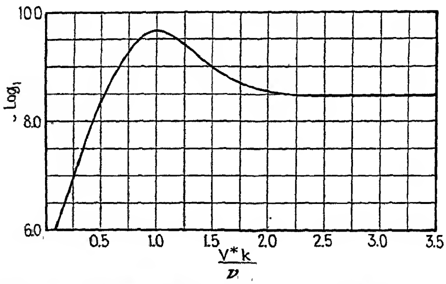

FIG. 63.—Generalized representation of velocity distribution *vs.* roughness.

In practice, many factors affect the velocity distribution. At inlet to a pipe, a certain distance is required for the velocity to reach an equilibrium distribution. Elbows and obstructions of all kinds disturb the distribution for some distance downstream.

Unequal roughness of the walls will result in an unsymmetrical velocity curve. These and other factors, as well as the disturbance of measuring instruments, will cause in practice deviations from the experimental curves mentioned above, which were obtained with more care than is generally practicable. However, these deviations should be regarded as such and not as evidence that the velocity distribution cannot be represented in an orderly manner.

62. Applications of Theory of Turbulent Flow.—There are many practical applications of the theory of turbulent flow, a few of which will be discussed. The student should note particularly that the mass transfer of fluid by turbulent mixing must be accompanied by the transfer of other properties of the fluid, such as temperature, dissolved and suspended material, and so forth. Therefore, two methods of attack are possible. One may study the nature of turbulence directly and reason from the acquired data to the rates of diffusion of these properties or one may measure the rates of diffusion and obtain information regarding the nature of turbulence.

One interesting effect of the velocity fluctuations in turbulent flow is illustrated by the pitot tube, which is used to measure the temporal average velocity. Assuming that the axis of the dynamic pressure opening is set parallel to the u-component of velocity, the instantaneous dynamic pressure is

$$p = \rho\frac{u^2}{2}$$

and the average pressure indicated by a sluggish pressure instrument connected to the tube is

$$\begin{aligned} p_{\mathrm{av}} &= \frac{1}{t}\int_0^t \rho\frac{u^2}{2}dt \\ &= \frac{1}{t}\int_0^t \rho\frac{\bar{u}^2}{2}dt + \frac{1}{t}\int_0^t \rho\bar{u}u'\,dt + \frac{1}{t}\int_0^t \rho\frac{u'^2}{2}dt \end{aligned}$$

The average value of the middle term is zero since $\bar{u}$ is a constant and therefore

Now $\frac{1}{2}\rho\bar{u}^2$ is the dynamic pressure corresponding to the true average velocity $\bar{u}$ and hence a pitot tube indicates a velocity in excess of the true average value when used in turbulent flow. If the tube is calibrated in a flow having the same degree of turbulence, the error would be eliminated, but this is not usually done. The error is in most cases small.

Another application is the prediction of the velocity gradient near a solid boundary but outside of the laminar layer. Starting from the expression for the tractive force,

inserting Eq. (4.37) gives as the differential equation

In this equation, κ is the universal constant of the turbulent interchange and τ is the tractive force at position z.

If τ is assumed to be constant, as in the case of steady flow without a longitudinal pressure gradient, integration between points z_1 and z, at which the velocities are $\bar{u}_1$ and $\bar{u}$, respectively, gives

$$\bar{u} - \bar{u}_1 = \qquad - \log_e$$

or

$$\frac{\bar{u} - }{V^*} \qquad (4.39)$$

The right-hand member of the second equation is the unknown function of Eq. (4.38) for the special condition τ = const.

A more important form of the expression for the velocity gradient is obtained if τ is assumed to vary with the distance z. In a circular pipe with symmetrical velocity distribution,

$$r - z$$

The corresponding equation for the velocity gradient in terms of the maximum velocity U and the radius r is

$$\frac{U - u}{V^*} = -\frac{1}{\kappa}\left\{\log_e\left[1 - \left(1 - \frac{z}{r}\right)^{\frac{1}{2}}\right] + \left[1 - \frac{z}{r}\right]^{\frac{1}{2}}\right\} \qquad (4.40)$$

This equation with $\kappa = 0.4$ appears in Fig. 62. Equation (4.40) should also apply to the velocity distribution in an open channel below the thread of maximum velocity if r is replaced by the

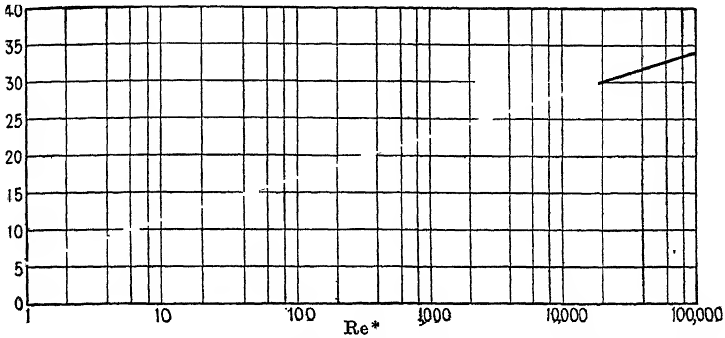

FIG. 64.—Generalized velocity distribution for smooth pipes. (*After Nikuradse.*)

depth. It is not valid near the center of a pipe ($z = r$), for there $d\bar{u}/dz$ must be zero if the velocity distribution is symmetrical and $d\bar{u}/dz$ continuous.

63. Smooth Pipes.—In Eq. (4.39), $u_1 \rightarrow -\infty$ as $z_1 \rightarrow 0$ and therefore this equation is not valid very close to the boundary.

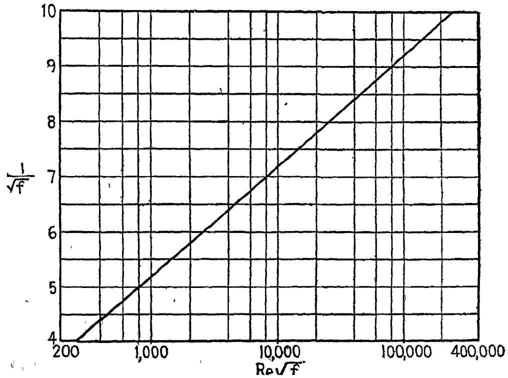

FIG. 65.—Relation of friction factor and Reynolds number for smooth pipes. (*After Prandtl.*)

However, u is probably a continuous function of z down to $z = 0$, the actual curve intersecting Eq. (4.39) at some point determined only by τ_0, ρ, and ν. Over a small range of z outside of this point

of intersection, τ may be considered as constant and equal to τ_0. The equation for the velocity distribution then reduces to

$$
\begin{aligned}
u &= \frac{1}{\kappa}\sqrt{\frac{\tau_0}{\rho}}\left(\log_e \frac{\sqrt{\frac{\tau_0}{\rho}}\,z}{\nu} + \text{const.}\right. \\
U^* &= \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{1}{\kappa}(\log_e \text{Re}^* + \text{const.})
\end{aligned}
\qquad (4.41)
$$

This formula was proposed by von Kármán in 1930. Figure 64 shows the data of Nikuradse for smooth pipes, plotted in accordance with Eq. (4.41). The complete equation with the empirical constants is

$$U^* = 5.5 + 5.75 \log_{10} \text{Re}^* \qquad (4.42)$$

The constant 5.75 corresponds to $\kappa = 0.40$.

In terms of the usual friction factor f and the mean velocity U_{mean}, Prandtl gives as the equation for smooth pipes over the whole range of Reynolds number,

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} (\text{Re}\sqrt{f}) - 0.8$$

This equation is represented in Fig. 65.

Equation (4.42) may be used to obtain some information regarding the thickness of the laminar sublayer by determining its intersection with Eq. (4.36).

$$
\begin{aligned}
U^* &= \text{Re}^* && (4.36) \\
U^* &= 5.5 + 5.75 \log_{10} \text{Re}^* && (4.42)
\end{aligned}
$$

Equating and solving for Re_0^*,

$$\text{Re}_0^* = 11.5 = \frac{z_0\sqrt{\frac{\tau_0}{\rho}}}{\nu}$$

which is very nearly the same as Eq. (4.33).

64. Rough Pipes.—The effect of the roughness is approximately as shown in Fig. 66, in which the velocities at all points are reduced by the same constant amount except near the wall. Considering pipes which are hydraulically rough, as defined by

Eq. (4.35) (fully developed turbulence and constant friction coefficient), Nikuradse has developed an equation to represent the roughness. Referring to Eq. (4.39), as $z_1 \rightarrow 0$, $u_1 \rightarrow -\infty$, and there is some point on this theoretical curve at which $u_1 = 0$. If it is assumed that this value of z_1 is proportional to the roughness and independent of τ_0, ρ, and ν,

$$z_1 = \alpha k$$

$$\left(\log_e \frac{z}{k} - \log_e \alpha\right) \tag{4.43}$$

Here, u is the velocity at any distance $z > z_1$, k is the size of the roughness and α is a constant factor. Using the velocity dis-

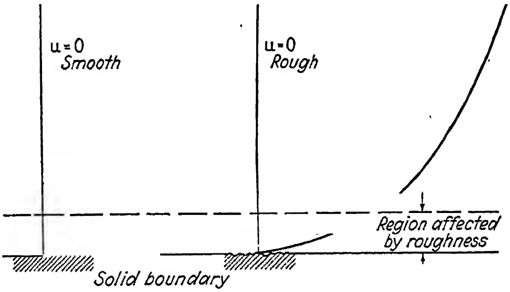

FIG. 66.

tributions for the experiments shown in Fig. 54, Nikuradse determined the constants in Eq. (4.42) as

$$u = V^*\left(2.5 \log_e \frac{z}{k} + 8.5\right.$$

The constant 2.5 corresponds to $\kappa = 0.40$. Changing the base of the logarithms,

$$u = V^*\left(5.75 \log_{10} \frac{z}{k} + 8.5\right.$$

The quantity M plotted in Fig. 63 is simply the constant $\log_e \alpha$ of Eq. (4.43). von Kármán's comment on the reliability of this generalized roughness equation has been mentioned previously.

The friction coefficient of *rough* pipes was found by Nikuradse to follow the equation

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} \frac{r}{k} + 1.74$$

On the basis of the experiments of both Fromm and Nikuradse, von Kármán obtained as the equation,

$$\frac{1}{\sqrt{f}} = 2.07 \log_{10} \frac{r}{k} + 1.50$$

The theory of turbulent flow has received some striking experimental verifications and the work in this field gives promise of ultimate success. However, it must be recalled that only the simplest situations have been analyzed. The pipes and channels met with in engineering practice present such a variety of roughness that, for the present, engineers must continue to use empirical methods. The greatest obstacle lies in the quantitative definition of hydraulic roughness in such terms as to apply to actual surfaces.

65. Skin Friction.—In marine engineering, aerodynamics, and other fields, it is important to be able to compute the frictional drag on a flat plate caused by a fluid moving parallel to it. The analytical methods employed in attacking this problem are in themselves of interest and the results will be employed later on so that the details of the analysis will be given. The treatment follows that of von Kármán.

The fluid stream is assumed to be so large that the effect of the solid boundary does not extend to the other boundaries, and that the pressure at a distance from the surface remains constant. Referring to Fig. 67, the leading edge of the plate is at $x = 0$,

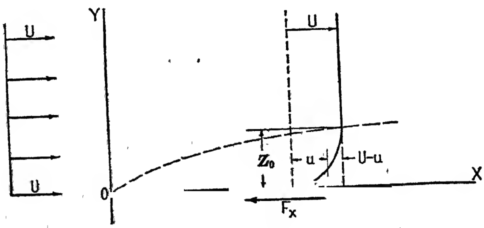

Fig. 67.

and the velocity at a distance upstream from the plate has a uniform value U. Since the pressure at a distance from the plate remains constant and the flow is approximately parallel to the plate, it is assumed that the pressure near the plate is also

constant. The rate of change of momentum must then be equal to the tractive force. Up to any point x, the total force is

$$- u)dz$$

where u is the velocity at distance z along a section perpendicular to the plate at point x, and b is the width of the plate. The shearing force per unit area at point x is evidently given by the equation

$$dF_x = \quad dx$$

In reality, the velocity at section x is found to differ from U by a negligible amount at a relatively short distance from the plate and the integration need only be carried to a finite distance z_0. The region in which the velocity is reduced by an appreciable amount will be referred to as the *boundary layer.*

Prandtl has analyzed laminar flow along a flat plate in the following manner: Along a section at any location x, the velocity distribution may be represented in general terms as

$$u = Uf(n), \qquad n = \frac{z}{z_0}$$

The function f must equal unity at $n = 1$ and zero at $n = 0$. Therefore the total frictional drag between $x = 0$ and x is

$$- f)dn = \alpha b \rho z_0 U^2$$

At the solid boundary, the tractive force is related to the velocity gradient by the basic equation

$$du \qquad U\left(\frac{df}{dn}\right. \qquad U$$

Here α and β are pure numbers having a significance defined by the equations themselves. From Eq. (4.44),

and

$$1 \ \beta_d$$

. Integrating,

$$(4.46)$$

Inserting the value of z_0 from Eq. (4.46) in Eq. (4.45) gives

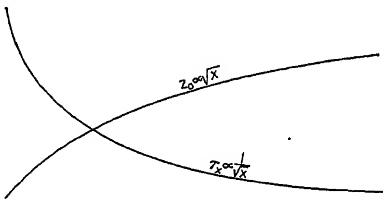

FIG. 68.

The total tractive force up to point x is

A local coefficient of resistance at point x may be defined by the equation

Here $(x\rho U/\mu)$ is a Reynolds number constructed by using x as the characteristic length. An average coefficient for the total tractive force up to point x is defined by

The mathematical theory gives 0.664 as the value of
The coefficients are then

$$0.664$$

$$C_f = \frac{1.33}{(\mathrm{Re}_x)^{\frac{1}{2}}}$$

Figure 68 shows schematically the variation of z_0 and τ_x along the plate.

Under certain conditions, the flow along a surface will be turbulent just as in the case of pipe flow. von Kármán has analyzed the problem as follows: The velocity distribution is assumed to be of the form

$$u = U\phi(n)$$

but the wall friction is assumed to follow the empirical relationship

$$\tau_x = \tfrac{1}{2}\rho U^2 \frac{\text{const.}}{\left(\dfrac{Uz_0}{\nu}\right)^m} \tag{4.50}$$

As for laminar flow,

$$F_x = b\int_0^z \rho u(U - u)dz = \alpha\rho U^2 z_0$$

$$\tau_x = \frac{1}{b}\frac{dF_x}{dx} = \alpha\rho U^2 \frac{dz_0}{dx}$$

Equating the two expressions for τ_x,

$$z_0{}^m\, dz_0 = \text{const.} \left(\frac{\nu}{TT}\right)^m dx$$

Integrating and solving for z_0,

$$z_0 = \text{const. } x\left(\frac{\nu}{Ux}\right)^{\frac{m}{m+1}} \tag{4.51}$$

Substituting for z_0 from Eq. (4.51) in Eq. (4.50) gives

$$\tau_x = \tfrac{1}{2}\rho U^2 \frac{\text{const.}}{(\text{Re}_x)^{\frac{m}{m+1}}} \tag{4.52}$$

In laminar flow $m = 1$, and comparison with Eq. (4.47) shows that the constant of Eq. (4.52) then becomes $(2\alpha\beta)^{\frac{1}{2}}$. Experiments on smooth circular pipes have shown that $m = \frac{1}{4}$ between $6000 < UD/\nu < 140{,}000$. Inserting this value and the numerical constant from the pipe experiments,

$$c_f = \frac{0.059}{(\text{Re}_x)^{\frac{1}{5}}}$$

Similarly, the coefficient of over-all resistance to a distance x from the leading edge is

$$C_f = \frac{0.074}{(Ux/\nu)^{1/5}}$$

Just as in the case of pipes, the analysis may be extended to include the effect of roughness. The transition from laminar to turbulent flow has been found experimentally to be dependent on a Reynolds number based on the thickness of the boundary layer. For the transition point,

$$\mathrm{Re}_c = \frac{Uz_0}{\nu} = 1600 \text{ to } 3500$$

Here z_0 is to be computed from Eq. (4.46) which applies to laminar flow. Figure 69 shows the conditions assumed to exist at the forward end of a flat plate. Empirical data on the friction of flat plates are included in Chap. VI.

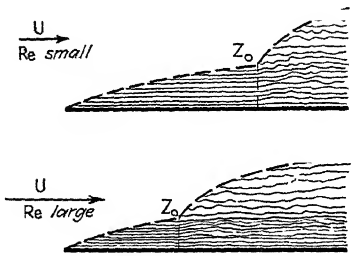

FIG. 69.

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Problems

1. Zerolene No. 3 at 90 deg. Fahr. flows between two very large parallel flat plates separated 1.0 in. If the mean velocity of the oil is 0.5 ft. per sec., what is the shearing stress at points 0.25 and 0.5 in. from the lower plate?

Ans. (*a*) $\tau_{0.25} = 0.033$ lb. per sq. ft.; (*b*) $\tau_{0.50} = 0$.

2. Derive an expression for the rate of discharge of a viscous fluid flowing in the annular space between two concentric circular pipes.

3. Derive an expression for the rate of discharge of a viscous fluid flowing down a plane of breadth b and inclined at an angle α if the depth is constant.

4. A capillary tube $\frac{1}{16}$ in. in diameter is 30 in. long. Under a head of 4.12 ft. of water, it discharges 0.41 lb. of water in 100 sec. What is the absolute viscosity of the water?

5. A circular glass tube 0.01 in. in diameter and 30 in. long carries water at 70 deg. Fahr. between two reservoirs under a differential head of 2.0 in. Considering the kinetic energy dissipated at discharge but neglecting entrance loss, compute: (*a*) the rate of discharge and average velocity in the tube, (*b*) the velocity head in the tube, (*c*) the power loss, and (*d*) the maximum velocity.

Ans. (*a*) 2.43×10^{-9} cu. ft. per sec., 4.46×10^{-3} ft. per sec.; (*b*) 6.18×10^{-7} ft.; (*c*) 2.52×10^{-11} ft.-lb. per sec.; (*d*) 8.92×10^{-3} ft. per sec.

6. A vertical glass tube 0.004687 ft. in diameter and 2.890 ft. long connects two reservoirs. A mercury differential manometer connected between the two reservoirs reads 8.69 in. when water at 77 deg. Fahr. flows from the upper reservoir to the lower. What is the rate of discharge, neglecting velocity head and entrance and discharge losses?

Ans. 12.12×10^{-5} cu. ft. per sec.

7. If the discharge in Prob. 6 was 5.87×10^{-5} cu. ft. per sec., what would be the average kinetic energy per unit weight?

Ans. 0.359 ft.-lb. per lb.

8. Two geometrically similar containers hold the same liquid and are rotated about vertical axes, the speed of the larger being 3600 r.p.m. If the scale ratio is 1:10 and time ratio 1:5, what is the rotative speed of the smaller in order to maintain dynamic similarity with respect to the centrifugal forces? What is the ratio of centrifugal forces?

Ans. (*a*) 18,000 r.p.m.; (*b*) 1:400.

9. The loss in head due to pipe friction is expressed $h_L = \phi(\nu, V, g, D, L, k)$ where h_L is feet of fluid, D is pipe diameter, L the length, and k the roughness measure. Change this equation to a form having a dimensionless grouping.

10. What are the dimensions of a shearing stress per unit of mass?

11. The orifice discharge coefficient can be expressed $C_d = f(Q, D_1, D_2, \mu, \rho)$, where 1 refers to pipe and 2 to orifice. Determine a dimensionless expression for C_d.

12. The resultant force F_R exerted on a stationary body held in an air stream is

$$F_R = \phi(\mu, \rho, V, L, k_1, k_2, V_c, u', \alpha)$$

where k_1 and k_2 are two linear dimensions measuring roughness, V_c is the velocity of sound and u' the fluctuating velocity in turbulent flow, α is the angle between the body and the direction of the air stream of velocity V, and L is a linear dimension. Change this equation to a form having a dimensionless grouping.

13. Oil of unit weight 56.30 lb. per cu. ft. and kinematic viscosity 0.00019 ft. sq. per sec. flows in a steel pipe of 6-in. diameter. What is the approximate velocity at which the flow changes from viscous to turbulent?

Ans. 0.76 ft. per sec.

14. Compute the quantity of water discharged through a 4-in. circular cast-iron pipe for a drop in the hydraulic gradient of 40 ft. of water between two points 1000 ft. apart. *Ans.* 0.452 cu. ft. per sec.

15. An oil-mercury manometer is used to determine the friction loss in a given length of the pipe of Prob. 13. If the manometer differential reading is 12.0 in., what is the head loss expressed in feet of oil? *Ans.* 14.07 ft.

16. In Prob. 13, what is the friction factor for the Weisbach equation if the mean velocity of the oil is 3.0 ft. per sec.? What is the head loss per 100 ft. of pipe when the oil flows with $V_m = 3.0$ ft. per sec.?

Ans. (*a*) $f = 0.0392$; (*b*) 1.096 ft. of water.

17. Castor oil is to be pumped through an 8-in. steel pipe at a velocity of 6 ft. per sec. and a temperature of 50 deg. Fahr. What will be the pressure drop and horsepower lost per 100 ft. of pipe?

Ans. (*a*) 15.09 lb. per sq. in.; (*b*) 8.28 hp.

18. Castor oil is pumped through a circular pipe of $1\frac{1}{2}$-in. diameter. If the temperature is 64 deg. Fahr. and the pipe is 50 ft. long, what is the power required to overcome the frictional resistance in pumping 1.8 cu. ft. per min.?

Ans. 19.65 hp.

19. Zerolene No. 3 is to be pumped through a 1-in. line 10 ft. long at a temperature of 90 deg. Fahr. The differential reading of an oil-mercury manometer is 15.72 in. How many pounds per minute of oil will be pumped?

Ans. 230 lb. per min.

20. What will be the delivery of oil in Prob. 19 if the temperature becomes 150 deg. Fahr. and the differential head expressed in feet of oil is the same?

Ans. 285 lb. per min.

21. Compute the diameter of wrought-iron pipe necessary for discharging 1500 gal. per min. of water, the pipe being 1000 ft. long and its discharging end 4 ft. lower than the surface of the reservoir supplying it. Neglect entrance loss. Assume water temperature of 68 deg. Fahr.

Ans. 13 in.

22. Find the pressure drop required to transmit 2400 lb. of castor oil per minute through a 5-in. cast-iron pipeline 1000 ft. long at 80 deg. Fahr. What is the power required to overcome the friction losses?

Ans. (*a*) 75.3 lb. per sq. in.; (*b*) 13.29 hp.

23. A 12-in. asphalt-coated cast-iron pipeline is 1000 ft. long and contains one 90-deg. elbow. When water at 77 deg. Fahr. is pumped through the line with a velocity of 3.20 ft. per second, the differential head is 4.00 ft. of water. How much of this loss is due to the added resistance of the elbow?

Ans. 0.43 ft.

24. 0.0795 lb. per sec. of dry air flows through a horizontal round brass pipe of 0.932 in. internal diameter and 24.0 ft. in length. The upstream pressure is 6.5 in. of mercury and temperature 89 deg. Fahr. Atmospheric pressure is 29.73 in. of mercury. The downstream pressure is 2.15 in. of mercury and temperature 88.2 deg. Fahr. Compute the friction coefficient and compare with accepted results. *Ans.* $f = 0.0182$.

25. 0.0913 lb. per sec. of dry air flow through a horizontal square brass pipe of sides 0.932 in. and length 24 ft. Upstream pressure is 34.73 in. of mercury absolute and temperature 90 deg. Fahr. Assuming isothermal

flow, compute the velocities at both ends of the section. Calculate the magnitude of the pressure drop by the (*a*) isothermal and (*b*) hydraulic equations.

Ans. $V_1 = 177.8$ ft. per sec.; (*a*) 1.61 lb. per sq. in.; (*b*) 1.77 lb. per sq. in.

26. Find the pressure drop necessary to transmit 3000 cu. ft. per min. of air at standard conditions through 10,000 ft. of 7-in. riveted pipe if the initial pressure is 120 lb. per sq. in. Assume constant temperature of 60 deg. Fahr.

27. Measurements of velocity distribution in a stream of air gave 36 ft. per sec. at 1.5 in. from a solid boundary and 35.5 ft. per sec. at 1.4 in. If the mixing length at 1.45 in. from the wall is 0.058 ft., find (*a*) the mechanical viscosity and (*b*) the shearing stress per unit of mass. Be careful to include the dimensions of your answer. Assume air at standard conditions.

Ans. (*a*) 4.8×10^{-4} lb. sec. per sq. ft.; (*b*) 12.1 ft.^{2} per sec.^{2}

28. It is desired to determine experimentally the mixing length by measurements of velocity distribution and pressure drop over length L of a conduit of narrow rectangular section, *i.e.*, $D << B$. Determine the equation for the mixing length in terms of the conduit dimensions and the experimental measurements.

29. Measurements in an air stream show a shearing stress per unit mass of 10 ft.^{2} per sec.^{2} at a distance of 1.35 in. from a wall. At 1.40 in. from the wall the velocity is 35.5 ft. per sec. and at 1.30 in. it is 29.9 ft. per sec. Determine (*a*) the mixing length and (*b*) the mechanical viscosity. Include the dimensions in your answer. Assume air at standard conditions.

Ans. (*a*) 0.00401 ft.; (*b*) 3.73×10^{-5} lb. sec. per sq. ft.

30. A pitot tube calibrated in a steady fluid stream with negligible fluctuating velocities was found to have a coefficient of unity, *i.e.*, $p = \frac{1}{2}\rho\bar{u}^2$. If the tube is to be used to determine the mean velocity in a fluid having an average fluctuating component of $0.2\bar{u}$, (root mean square value), by what coefficient must the pressure be multiplied in order to calculate the true mean velocity from actual readings? *Ans.* 0.961.

31. A thin flat plate is 10 ft. long and 24 in. wide. What is the total skin friction of the plate if the undisturbed air velocity is 200 ft. per sec.? If the critical Reynolds number is 2000, compute the distance to the point at which the flow changes from laminar to turbulent. Compute the local friction coefficient 6 in. from the leading edge. Assume air at standard conditions.

CHAPTER V

WEIRS, ORIFICES, AND GATES

In engineering practice it is often desirable to use for metering or control of flow, devices which establish a unique relationship between the rate of discharge and a measurable variable connected with the total head. In some devices the elevation of the energy grade line is used, while in others a change in the elevation of the hydraulic grade line is employed. Broad- and sharp-crested weirs, nozzles, orifices, bottom outlets, spillways, venturi meters, and many other structures and devices fall in this category. The theoretical and experimental work on this subject has been so extensive that only a review of the more elementary principles will be attempted here. For detailed information on the characteristics of particular devices, the reader should consult the original reports or one of the engineering handbooks.

Structures and devices which are used for metering or control may be divided on the basis of the underlying theory into three groups as follows:

1. *Free Discharge.*—The discharging jet passes into a region of negligible density in which the pressure may be considered as constant (orifices and weirs discharging liquid into air).

2. *Submerged Discharge.*—The discharging jet passes into a region filled with the same fluid, or a fluid of comparable density, in which the vertical pressure gradient is linear (orifices discharging water into water or air into air).

3. *Semisubmerged Discharge.*—Applies only to discharge of liquids in which one side of the jet is in contact with a region of negligible density, while the other is in contact with a solid surface or a liquid. The pressure is determined not by that of the surrounding region alone but by the depth and curvature of the jet itself (spillway discharging water).

Except where otherwise specifically stated, all equations in this chapter apply to steady flow.

66. Free Discharge of Frictionless Liquids.—Since gases diffuse to fill any space in which they are placed, free discharge is

limited to liquids. The derivation of the basic equations proceeds directly from Bernoulli's energy equation and empirical coefficients are inserted to reconcile theory and experiment.

To obtain the general equation of free discharge, consider the flow through an irregular opening (Fig. 70) in a vertical plate which forms a portion of one side of the supply reservoir. Assume also that there is no frictional loss and that the upper surface of the reservoir is under the same pressure as the discharging stream.

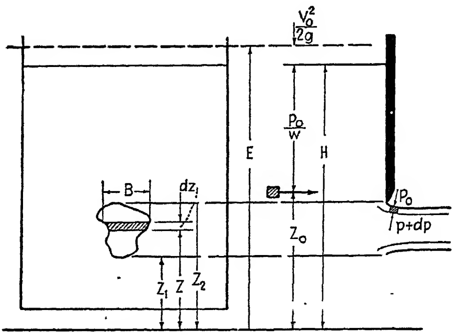

FIG. 70.

The velocity of efflux at elevation z is then by Bernoulli's equation

$$V = \sqrt{2g(E - z)}$$
$$E = \frac{p_0}{w} + z_0 + \frac{V_0^2}{2g}$$

This is not, however, the velocity in the plane of the opening since, in general, a certain amount of contraction occurs because of the inertia of the particles moving in from the sides. In the case of a sharp-edged orifice or weir, the conditions in the plane of the opening are as shown in Fig. 70 where it is evident that the pressure in the interior of the jet is greater than at the boundaries except for the particles moving in parallel paths. This latter point is called the *vena contracta* or contracted section and it is here that the velocity is known. Unfortunately the area at the vena contracta cannot be deduced readily and consequently we have reached an impasse in that the area and corresponding velocity are not known at any section.

In the usual derivation of the "theoretical" equations for free discharge, it is assumed that the velocity, $\sqrt{2g(E - z)}$, exists over the entire width of the opening and is perpendicular to it. The inaccuracy of this assumption is shown by the fact that a corrective factor, sometimes as low as 0.59, must be applied to make the equations agree with experiment. However, this method gives the correct form of the equation, and the corrective factors for any one liquid are approximately constant, so that, in spite of being 40 per cent in error, the results are useful.

On the basis of the above assumptions, the general equations are

$$V = \sqrt{2g(E - z)}$$
$$dQ = BV\,dz$$

where $B = f(z)$

$$Q = \int_{z_1}^{z_2} f(z)\sqrt{2g(E - z)}\,dz$$

The datum is usually taken at the lowest point of a weir and at the center of an orifice of regular form, conditions which simplify the equations but are not necessary.

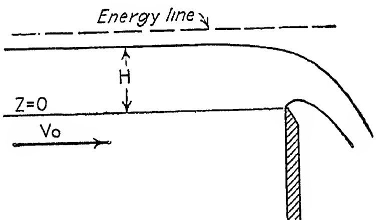

Fig. 71.

67. Sharp-crested Weirs. Theory.—The term *weir* is generally restricted to structures causing free discharge of a liquid in such a manner that the upper surface remains continuously in contact with the surrounding fluid of negligible density, which in most instances is air. A sharp-crested weir is one which has a sharp-edged aperture or crest from which the escaping stream or nappe springs clear. The use of weirs as metering devices is in part due to their simplicity of construction and this consideration should not be lost sight of in appraising the relative advantages of different proposed weir forms. Straight and circular

outlines are the simplest forms for machining and other shapes should be distinctly advantageous from other standpoints to warrant the added cost of fabrication.

Figure 71 represents the general shape of the discharging nappe for sharp-crested weirs of all shapes. A lateral contraction also

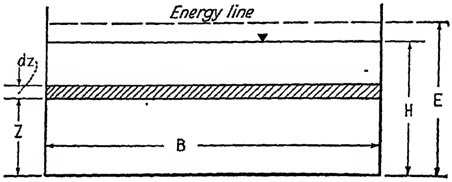

Fig. 72.

occurs if the approach channel is wider than the weir opening. Taking the crest of the weir as the datum for the energy gradient,

$$E = H + \frac{V_0^2}{2g}$$

$$Q = \int_0^H f(z)\sqrt{2g(E - z)}\, dz \tag{5.1}$$

To illustrate the derivation of equations for particular shapes, Eq. (5.1) will be applied to two of the more common forms.

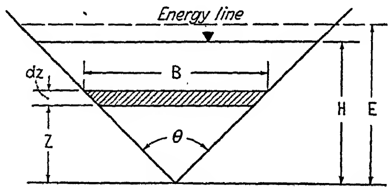

Fig. 73.

1. *Rectangular Weir.*—Referring to Fig. 72,

$$f(z) = B = \text{const.}$$

$$Q = \int_0^H B\sqrt{2g(E - z)}\, dz = -\sqrt{2g}B\tfrac{2}{3}\Big[(E - z)^{\frac{3}{2}}\Big]_0^H$$

$$= \frac{2}{3}B\sqrt{2g}\left[\left(H + \frac{V_0^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{V_0^2}{2g}\right)^{\frac{3}{2}}\right] \tag{5.2}$$

If the approach velocity, V_0, is negligible, the equation reduces to

$$Q = \tfrac{2}{3}\sqrt{2g}BH^{\frac{3}{2}}$$

2. *Triangular Weir.*—Referring to Fig. 73, the width varies with the elevation above the crest. The quantity $f(z)$ is then

$$f(z) = 2z \tan \frac{\theta}{2}$$

Inserting this relationship in Eq. (5.1) and integrating, gives

If the velocity of approach is negligible, this equation reduces to

$$Q = \frac{8}{15} \tan \frac{\theta}{2} \sqrt{2g} H^{\frac{5}{2}} \qquad (5.5)$$

The same procedure is followed in deriving the theoretical equations for other weir forms. The two necessary conditions are

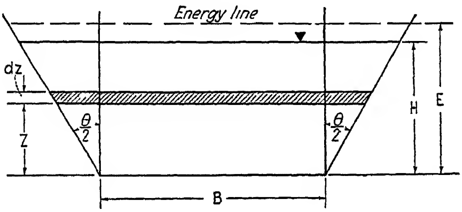

FIG. 74

that the variable width be expressed as a function of z and that the resulting equation be integrable. The equations for weir shapes made up of several simple forms may be obtained by direct addition as in the case of the trapezoidal or Cippoletti weir shown in Fig. 74. The theoretical equation is the sum of the equations for the rectangular and triangular weir. Considering an arrangement in which the velocity of approach is negligible,

$$Q = \frac{8}{15} \tan \frac{\theta}{2} \sqrt{2g}$$

68. Sharp-crested Weirs. Experimental Results.—Measured discharges are roughly 60 per cent of the theoretical values and the corrective coefficient does not vary widely. The form of the equation is very nearly representative of experimental data. The causes of the quantitative disagreement are the following:

1. *Contraction of the Nappe after Passing the Plane of the Weir.*—The lower surface of the escaping nappe is tangent to the weir face and the free surface must drop as the velocity increases. Figure 75 shows the magnitude of the contraction for a rectangular weir in a wide channel. The amount of the lateral contraction shown applies only when B is large in comparison with H.

2. *Character of the Upstream Face.*—A rough surface slightly increases the friction loss but tends also to reduce the contraction to a greater degree which results in a greater discharge than with a smooth face.

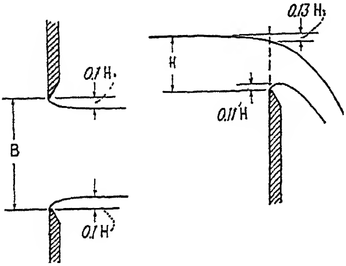

FIG. 75.

3. *Viscosity.*—Viscous resistance along the weir face tends to reduce the lateral velocities and the contraction, and an increase in viscosity results in a greater discharge.

4. *Adhesion and surface tension* affect the discharge at low heads and prevent it altogether below a certain minimum value of H which depends upon the shape of the weir.

5. *Pressure below the Nappe.*—Where the nappe is confined between solid boundaries, free access of air to the space below the nappe is prevented. Air is then gradually removed resulting in a negative pressure and an increase in discharge at a constant value of H.

6. *Dimensions of Channel of Approach.*—The derivation of the theoretical equation included the effect of the velocity of approach on the velocity of the nappe but did not include its effect on the contraction. If the area of the approach channel is small, only a fraction of the discharge enters the weir opening laterally and the contraction of the jet is reduced. Not only the total area but also the ratio of width to depth is important.

7. *Distribution of Velocity of Approach.*—In the derivation of the theoretical equations, the approach velocity V_0 was assumed to be the same for all of the liquid. A nonuniform distribution may materially affect the discharge.

Before considering the empirical formulas for weirs, it is important to consider the importance of retaining the term $V_0^2/2g$

in the theoretical equations [Eqs. (5.2) and (5.4)]. The velocity of approach depends upon the discharge and the area of the channel which in turn depends upon the shape of the channel and the head. It appears then that V_0 is not an independent variable. If by retaining $V_0^2/2g$ in the theoretical equations, the corrective coefficient obtained by experiment became a constant for each shape, the complete form would be worth using, but this is not the case since the effect of the velocity of approach on the contraction was not taken into consideration. Since the corrective coefficient is of the order of 0.6, simplified equations, such as Eqs. (5.3) and (5.5), are adequate. The entire effect of the velocity of approach then appears in the coefficient.

Considering the general equation first, the actual discharge may be represented as

$$Q_a = C\int_0^H f(z)\sqrt{2g(H - z)}\, dz$$

Here, C is a coefficient which varies in such a manner as to make the equation fit the experimental data and Q_a is the actual discharge. The velocity of approach has been eliminated by replacing E by H. The formulas for rectangular and triangular weirs become

(rectangular) (5.6)

$$Q_a = C\frac{\circ}{15}\tan\ 2 \qquad \text{(triangular)} \qquad (5.7)$$

Similar equations apply to the other forms for which the theoretical equations can be integrated. The following empirical equations are typical of the equations used in practice.

a. Triangular Weir.—The first experiments on a triangular weir were made by Thomson using a 90-deg. notch. He obtained the equation $Q_a = 2.536H^{\frac{5}{2}}$. This equation corresponds to $C = 0.59$. The number of significant figures in his equation is not justified by the experimental accuracy. Experiments by Cone on a series of triangular weirs of different angles showed that the equation could be represented as $Q = KH^n$. The values of K and n and the value of C at $H = 1$ ft. were as follows:

Angle of notch (2θ), deg.	K	n	C ($H = 1$)
120	4.40	2.49	0.59
90	2.49	2.48	0.58
60	1.45	2.47	0.59
30	0.68	2.45	0.60
$28° - 4'$	0.64	2.44	0.60

b. Rectangular Weir.—When the width of the approach channel equals the length of the weir crest, lateral contraction does not occur. Spreading of the nappe may be prevented by continuing the side walls a short distance downstream. Under such conditions the discharge is very nearly proportional to the length of the crest, the only deviation being due to the slight friction of the walls. For such conditions Rehbock has obtained the empirical equation,

$$C = 0.605 + \qquad 0.08\frac{H}{z}$$

where H is the head in feet and z is the height of the crest above the bottom of the channel. The second term affects the value of C at low heads only and apparently results from molecular forces.

Rehbock's data may also be represented by the equation

$$Q_a = B\left(3.22 + 0.455\frac{H}{z}\right)(H + 0.004)^{\frac{3}{2}}$$

This equation is dimensionally correct and seems to show that the measured head is increased by 0.004 ft. which may represent a constant force of capillarity or surface tension tending to draw the water over the weir at all heads. The above equation is valid for all values of B, H, and z so long as the nappe is perfectly free and aerated from below and is not affected by backwater. Below $H = 0.0085$ ft. no discharge will occur. For

$$0.0085 < H < 0.191$$

the nappe may be free or clinging depending on sequence of heads. For $H > 0.191$ the nappe is always free. The relation between C and H is of the general form shown in Fig. 76.

For weirs in wide channels, lateral contractions occur. James B. Francis obtained the equation

$$Q_a = 3.33(B - 0.2H)\Big[\Big(H +$$

Assuming $V_0 = 0$, $B = 5$, and $H = 1$, which are in the range of these experiments, $C = 0.59$. These experiments of Francis have been reanalyzed in a number of ways. An example of the forms obtained is that of Barnes,

$$Q_c = 3.324\left(H + \frac{V_0^2}{2g}\right)^{1.49} B^{1.11}\Big[B + 2\Big(H$$

Formulas of such complexity are not justified by the accuracy of the original data and are of doubtful utility since one may

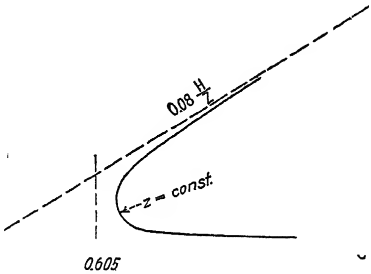

FIG. 76.—General representation of the new coefficient C vs. H as determined by Rehbock's formula $C = 0.605 + \frac{1}{305H} + 0.08\frac{H}{z}$.

consult the original tabulation of data for these or similar experiments if great accuracy is desired.

c. Trapezoidal Weir.—In the Francis formula for the rectangular weir with end contractions, the crest length is reduced by $0.2H$ to take into account the lateral contraction. Cippoletti thought that by sloping the sides and in effect adding a triangular weir having a discharge equal to $C(0.2)\frac{2}{3}\sqrt{2g}H^{\frac{5}{2}}$ he could reduce the empirical equation to the form of Eq. (5.6). Unless this result is obtained, there appears to be no advantage in using this type of weir. Experiments by Cone showed that the equation which best fitted a weir of this type was

$$Q_a = \qquad 0.609H^{2.5}$$

$$1 < B < 4\,\text{ft.}, \qquad 0.2 < H < 1.5\,\text{ft.}$$

d. Circular Weirs.—Circular weirs can be constructed easily and with greater accuracy than other types. They can also be placed more easily since they are symmetrical about every diameter. However, the theoretical equation for the discharge is difficult to obtain and so cumbersome as to be useless practically. Hence the discharge equations are empirical. One series of experiments gave

$$Q_a = 2.747H^{1.807}D^{0.693}$$

where H and D are in feet and Q is in cubic feet per second. The limits of application are

$$0.2D < H < D, \qquad 0.5 < D < 2.0 \text{ ft.}, \qquad H > 0.15 \text{ ft.}$$

Experimental results on a wide range of sizes have been published by F. W. Greve.

Weirs are relatively sensitive to changes in construction and operation and the safe procedure is to calibrate them in place. Failing facilities for calibration, the next best procedure is to find a series of experiments covering the range desired and to reproduce the experimental setup. Extrapolation of weir formulas beyond the range of the underlying experiments is dangerous, and if application is restricted to this range, it would appear to be preferable to use the data directly rather than through the medium of a formula.

69. Generalized Representation of Weir Coefficients.—Since the use of weirs has been confined largely to water at normal temperatures and under appreciable heads, the effects of surface tension and viscosity have not been important and not much has been done in the way of generalizing the experimental results. Considering weirs of the same shape in geometrically similar approach channels, the rate of discharge Q_a probably depends upon the head H, the width of the weir B, the gravitational force per unit mass g, the area of the approach channel A, the roughness of the weir face k, the density of the liquid ρ, the surface tension σ, and the viscosity μ. Following the methods of dimensional analysis

$$f(Q_a, H, B, g, A, k, \rho, \sigma, \mu) = 0$$

Three primary quantities are represented in these nine secondary quantities and there will be six dimensionless groups. The Π's may be obtained as follows:

$$\Pi_1 = Q_a{}^a g^b \rho^c H$$
$$[1] = L^{3a}T^{-a}L^bT^{-2b}M^cL^{-3c}L$$

$$\begin{aligned} \text{Mass:}\quad & 0 = c \\ \text{Length:}\quad & 0 = 3a + b - 3c + 1 \\ \text{Time:}\quad & 0 = -a - 2b \end{aligned} \qquad \begin{cases} c = 0 \\ b = +\frac{1}{5} \\ a = -\frac{2}{5} \end{cases}$$

Similarly,
$$\Pi_2 = \qquad\qquad \Pi_3 = \frac{}{Q_a^{\frac{2}{5}}},$$
$$\Pi_4 = \qquad\qquad \Pi_5 = \frac{\sigma}{\rho g^{\frac{3}{5}} Q_a^{\frac{4}{5}}}$$

These dimensionless groups may be changed to more familiar forms by recombinations. The principle involved is that similarity of flow will exist if all these groups have identical values simultaneously for two systems and the regrouping may be made on this basis. Starting with Π_1, if this group is raised to the three-halves power and multiplied by Π_2,

The quantity B is a general dimension representing the width of the opening and may be considered as the width at the water surface. The combination of Π_1 and Π_2 yielded a group differing from the coefficient of discharge by a constant factor. By similar combinations, the equation may be reduced to

$$C = \phi\left(\frac{B}{\;},\qquad k \qquad \frac{}{\rho H^{\frac{3}{2}}}\right)$$

where C is the coefficient of discharge. There is nothing essential about these particular forms and others may be obtained by different combinations. The first three terms merely represent the geometrical similarity including the roughness. The other two are Weber's number and Reynolds' number. Eisner has analyzed data on sharp-crested, rectangular weirs without end contractions by plotting $\frac{E}{\sqrt{\sigma/w}}$ against $C'\frac{E}{E+p}$ and drawing lines of constant C'. Here, E is the height of the energy gradient

above the weir crest and a linear dimension which may therefore replace H in the above equations, C is the coefficient of discharge derived from $C' = Q_a/B\sqrt{2g}E^{\frac{3}{2}}$, and w is the weight per unit volume. The analysis would be changed in no respect if C' were substituted for C in the dimensional equations above. Eisner has merely substituted other ratios and has combined them in a way which facilitated plotting the data. He does not represent the effect of viscosity.

For triangular weirs, Smith writes a Reynolds number in the form VH/ν in which he substitutes $V = \sqrt{2gH}$ yielding $H/\nu^{\frac{2}{3}}$ as a criterion. The weight per unit mass, g, has been neglected since it is constant. The discharge equation used to define the coefficient is $Q = CH^{\frac{5}{2}}$. Substituting for H in the criterion, $C = f(Q/\nu^{\frac{5}{3}})$. Smith gives a curve of C as a function of $Q/\nu^{\frac{5}{3}}$, which appears to be based only on measurements at a constant value of ν. Since the measured variations in C may be partly the result of surface tension effects, this curve can not be regarded as a satisfactory method of accounting for the effect of viscosity unless it is substantiated by data covering a wide range of viscosity.

70. Effect of Small Waves on Weir Discharge.—It is commonly observed that the surface of open bodies of water is disturbed by wind waves or by long-period surges which may be expected to affect the discharge. A. H. Gibson has developed a theoretical equation for this condition which is based on the assumption that the instantaneous discharge during the passage of a wave is the same as would occur during steady discharge with the same depth above the crest. For sinusoidal waves, he found on this assumption that the discharge, Q_w, over a 90-deg. triangular weir with waves increases over that without waves, Q_a, in the ratio $\dfrac{Q_w}{Q_a} = 1 + 0.94\dfrac{a^2}{H^2}$, where H is the average head and $2a$ is the height of the wave from trough to crest. For a sharp-crested rectangular weir he found

$$\frac{Q_w}{Q_a} = 1 + 0.19\frac{a^2}{H^2}$$

These ratios were confirmed by experiments using short-period waves with heights equal to or less than the average head on the weir. Gibson concludes that "with waves whose height does not

exceed the mean head on the weir-crest, and with a normal type of measuring weir, whose coefficient under normal conditions is known, a correction may be applied which will enable the effect on the discharge to be computed within 1 per cent."

These experiments have considerable practical significance. They indicate that with less rapid variations in head whether periodic or progressive, the instantaneous rate of discharge may safely be assumed to be related to the instantaneous head by the same equation that applies to steady flow. They also provide a basis for establishing a "tolerance" for surface disturbances during weir measurements and this is probably their greatest value since correction of weir data for wave height would be

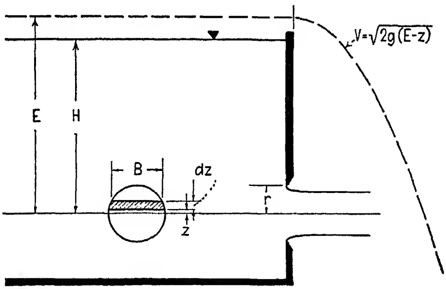

FIG. 77.

regarded by most engineers with suspicion. For example, to limit the effect of waves to 0.1 per cent for a rectangular weir, the ratio of wave height to head should be less than 0.15, which shows that the surging normally observed in weir boxes is of negligible effect. Similar limits for each type of weir might be incorporated into test codes.

71. Theoretical Equations for Orifices and Nozzles.—The basic equations and the simplifying assumptions are the same for vertical orifices and nozzles discharging frictionless liquids as for weirs. Considering the case of a circular, sharp-edged orifice in a vertical plate (Fig. 77), the theoretical velocity at the vena contracta will vary with elevation according to the relationship

$$V = \sqrt{2g(E - z)}$$

Here, z is measured above the center of the orifice. Assuming as

before that discharge occurs over the entire area of the opening,

$$\overline{\quad - z)}dz$$

Since the opening is circular, $B = 2\sqrt{r^2 - z^2}$ and

$$Q = 2\int_{-r}^{+r} \sqrt{2g(E - z)}\sqrt{r^2 - z^2}\, dz$$

Expanding, simplifying, and integrating, we obtain

$$Q = \pi r^2 \sqrt{2gE}\left(1 - \frac{r^2}{32E^2} - \frac{5r^4}{1024E^4} + \cdots\right)$$
$$= A_0 M \sqrt{2gE}$$

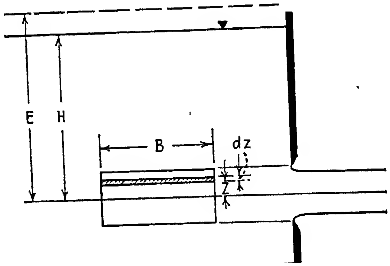

FIG. 78.

where A_0 is the area of the orifice and M represents the summation of the series. The value of the multiplier M is

$E/2r = E/d$	M
0.5	0.960
0.8	0.987
1.4	0.996
3.0	0.999

It is evident that for heads which are more than three times the diameter, the discharge can be based upon the head at the center of the opening and the area.

Incidentally, in the above expression for Q, the series is not convergent when E is less than r, which is the condition when the aperture functions as a weir. Consequently the discharge equation for a circular weir cannot be obtained by direct integration.

The discharge equation for a rectangular orifice in a vertical flat plate is obtained in the same manner with B = const.

$$V = \sqrt{2g(E - z)}$$

$$Q = \int_{-\frac{z_0}{2}}^{+\frac{z_0}{2}} 2g\sqrt{E - z}B\,dz$$

$$Q = \sqrt{2g}B\frac{2}{3}\left[\left(E - \frac{z_0}{2}\right)^{\frac{3}{2}} - \left(E + \frac{z_0}{2}\right)^{\frac{3}{2}}\right]$$

Expanding and simplifying,

$$Q = Bd\sqrt{2gE}\left(1 - \frac{2}{192}\frac{z_0^2}{E^2} + \cdots\right) = BD\sqrt{2gE}M$$

where M has the following values:

E/z_0	M
1.0	0.99
3.0	0.999

In the case of an orifice in a horizontal plate, all of the liquid has the same theoretical velocity and the theoretical discharge equation for all shapes is

$$Q = A_0\sqrt{2gE}$$

where E is to be measured above the vena contracta and not above the plane of the orifice.

A nozzle is merely an orifice with an added discharge section which confines the stream and causes the vena contracta to have the same area as the discharge end of the nozzle itself. For liquids the tip of a nozzle generally has parallel sides and the longitudinal outline should conform approximately to the shape of the jet from an orifice. The equations obtained for orifices are then applicable to nozzles but the corrective coefficients will differ, since the contraction is zero within the nozzle itself and the coefficient of contraction is unity.

72. Orifice and Nozzle Coefficients for Liquids.—When $E > 3d$, the discharge can be represented by the equation

$$Q = C_dA_0\sqrt{2gE} \tag{5.8}$$

where A_0 is the area of the orifice and C_d (the coefficient of discharge) is a corrective coefficient which includes both the effect of contraction and of friction. To separate these effects, write

$$A_j = C_c A_0$$

where A_j is the area of the jet at the vena contracta and C_c is the coefficient of contraction. Because of the friction, the average velocity at the vena contract is less than the theoretical value and is

$$V_j = C_v \sqrt{2gE}$$

where C_v is less than unity. From continuity,

$$Q_a = A_j V_j = C_c C_v A_0 \sqrt{2gE}$$

and consequently,

$$C_d = C_v C_c$$

Expressing the loss of mechanical energy due to friction as a fraction of the velocity head at the vena contracta,

$$V_j^2 \qquad \text{or} \qquad V_j =$$

The coefficient of velocity is therefore related to the friction coefficient by the equation

$$C_v = \frac{1}{\sqrt{1 + K}}$$

The mechanical efficiency of a nozzle or orifice is the ratio of the usable mechanical energy in the jet to the total mechanical energy expended, or

$$e_m = \frac{\text{power delivered}}{\text{power expended}} = \frac{Q_a \dfrac{V_j^2}{2g}}{Q_a E} = C_v^2 = \frac{1}{1 + K}$$

For very smooth orifices and nozzles discharging water, the coefficient of velocity ranges from 0.97 to 0.99 with 0.98 a usual value.

The coefficient of contraction of a perfect fluid passing through a circular orifice has been considered previously in connection

with momentum changes. Hydrodynamical studies show that the contraction of a perfect liquid passing through a long narrow slit or a circular opening is

$$C_c \qquad \pi + 2 = 0.611$$

a value which agrees fairly well with experiments on water at ordinary temperatures.

The coefficients of contraction, velocity, and discharge will vary with the character of the liquid, the head, the diameter and roughness of the nozzle or orifice, and the area of the approach channel, or more precisely, for geometrically similar boundaries, these coefficients will vary with the Reynolds number. Figure

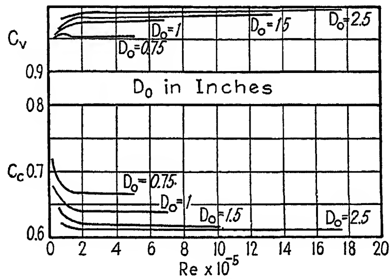

FIG. 79.—Coefficient of velocity and contraction for circular sharp-edged orifices as a function of Reynolds number.

79 shows the variation of coefficients of contraction and velocity with Reynolds number as obtained experimentally by Smith and Walker for circular sharp-edged orifices with free discharge. The coefficient of discharge is the product of the two coefficients shown.

As in the case of weirs, it is convenient to simplify the discharge equation by replacing E by the height of the hydraulic grade line, h, with the difference that h is to be measured above the center of the opening. The equation is then

$$Q_a = CA_0\sqrt{2gh} \tag{5.9}$$

where the coefficient C includes the effect of the velocity of approach both on the velocity of the jet and on the contraction. Even when Eq. (5.8) is used in place of Eq. (5.9), the coefficient C_d varies with the velocity of approach and it is therefore more

convenient and equally accurate to use the simpler form with a proper coefficient for each approach condition.

The coefficient of discharge includes the effect of both friction and inertia and is therefore generally used. An added advantage is that it can be measured easily since it is obtained from Eq. (5.9), as

$$C = \frac{Q_a}{A_0\sqrt{2gh}}$$

which requires only measurements of the rate of discharge and the pressure head.

Using the data of Moir, Unwin found that for water the coefficients of discharge of smooth, sharp-edged circular orifices *with full contraction* can be represented by the equation

$$C_d = 0.6075 + \frac{0.0098}{h^{\frac{1}{2}}} - 0.0037D$$

temp. = 51 to 55 deg. Fahr.
h = head, feet.
D = diameter, feet.

This equation probably carries more significant figures than are warranted. A formula proposed by T. P. Strickland for the same conditions is

$$C_d = 0.5925 + {}^{0.018}$$

The formula proposed by Barnes as a result of a study of the available experimental data is

$Q =$
$h > 2$ ft.
Q = capacity, cubic feet per second
h = head, feet
d = diameter, inches

It should be noted that the coefficient for full contraction is here roughly 0.60.

If the area of the channel of approach is small as compared with the area of the orifice, the forward motion of the liquid directed towards the orifice itself is but little influenced by the

radial motion of the liquid from the edges of the orifice and the contraction is incomplete. Figure 80 shows schematically the arrangement of an International Standard orifice modified for use discharging freely from the end of a pipeline. The pressure connection is an annular ring at the junction of the pipe wall and the orifice plate. Tests made with 8-in. pipe showed that the

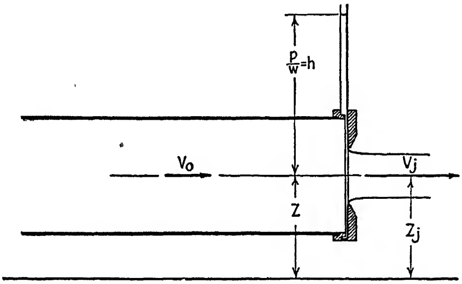

FIG. 80.—Schematic representation of International Standard orifice (see Fig. 41).

coefficient was independent of the Reynolds number but varied with the ratio of orifice diameter to pipe diameter as follows:

Ratio of orifice to pipe diameter	Coefficient C (30 diameter approach)
0.40	0.61
0.60	0.65
0.80	0.76

The effect of surface tension is appreciable only at very small diameters. It causes the pressure in the jet to exceed atmospheric pressure and therefore reduces the velocity but it also causes a decrease in the contraction and the net effect is an increase in discharge.

SUBMERGED DISCHARGE

73. General Equations for Liquids.—Just as for free discharge, the jet contracts (either freely or as constrained by boundaries) but the region into which the discharge occurs is sufficiently large that the pressure varies hydrostatically and the jet is not under a uniform pressure. No surface tension effect is involved.

Referring to Fig. 81, the velocity of the jet at any elevation, z, is $V_z = \sqrt{2g(E_1 - h_2)}$ and is constant over the entire jet. The rate of discharge is

$$Q = \int_{-z_0}^{+z_0} BV_z\, dz = \sqrt{2g(E_1 - h_2)} \int_{-z_0}^{+z_0} f(z)\, dz$$
$$Q = A_2\sqrt{2g(E_1 - h_2)}$$

The same equation applies with the axis of the orifice vertical or inclined. The actual discharge is represented by the equation

$$Q_a = C_d A_2 \sqrt{2g(E_1 - h_2)} \tag{5.10}$$

where C_d is a coefficient of discharge having a value about 1 per cent greater than for free discharge under otherwise identical

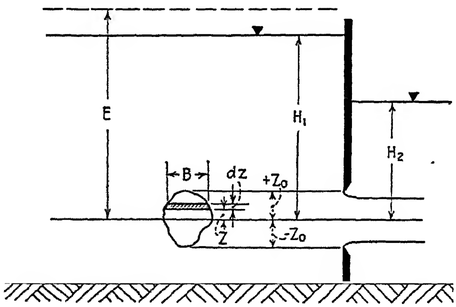

FIG. 81.

conditions. This increase in coefficient probably results from the dragging effect of the surrounding fluid, which would retard the outer layers of the jet and reduce the contraction. The equations for liquids apply with sufficient accuracy to gases if the pressure ratio is near unity. At atmospheric pressure the effect of compressibility is negligible up to differential pressures of about 5 in. of water, and the remarks on liquids will apply to gases within this range.

In Eq. (5.10), the quantity E_1 includes the velocity of approach. It is open to the same objections as the inclusion of the velocity of approach in weir formulas unless the extent of the contraction is known as is the case with the venturi meter and the flow nozzle. In Eq. (5.10),

$$h_2 = \frac{p_2}{w} + z_2$$

and

$$E_1 = \frac{p_1}{w} + z_1 + \frac{V_1^2}{2g}$$

for the general case of an inclined pipe. Inserting these values and the continuity relationship, Eq. (5.10) becomes

$$Q_a = C_d A_2 \sqrt{\frac{2g\Delta h}{1 - \left(\frac{A_2}{A_1}\right)^2}} \tag{5.11}$$

where $\Delta h = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)$, which is the quantity measured by a differential manometer connected between areas A_1 and A_2. Equation (5.11) has a number of alternative forms. It will be recognized as the equation applicable to the venturi meter derived in Chap. III. It is merely an application of Bernoulli's equation corrected for the effect of friction losses and contraction of the jet.

The appearance of the gravitational force per unit mass, g, is of some interest in connection with the conditions for similarity since it appears to indicate that gravity is a controlling factor and that the Froude law of similarity should apply as well as that of Reynolds. However, for liquids in horizontal flow the acceleration of gravity appears only as a conversion factor and one may replace $g\Delta h$ by means of the relationship

$$\frac{w\Delta h}{\rho} = \frac{\Delta p}{\rho}$$

giving

$$Q_a = C_d A_2 \sqrt{\frac{2\Delta p}{\rho\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]}} \tag{5.12}$$

Equation (5.12) is perhaps preferable. The quantity g entered Eq. (5.11) only because it has been found convenient to express head in feet of the fluid flowing rather than the pressure difference.

Since friction and inertia are the basic factors involved in the similarity of flow in geometrically similar systems without a free

surface, the conditions for similarity of flow of liquids through orifices, nozzles, and similar devices may be written as

$$C_d = f\left(\frac{VD}{\cdot}, \frac{A_1}{A}, \frac{k}{D}, \alpha_1, \qquad\qquad , \frac{l_1}{D}, \frac{l_2}{D}\right.$$

Here k/D expresses the relative roughness of both meter and approach pipe and α_1, α_2, . . . and l_1/D, l_2/D . . . are a series of angles and length ratios necessary to specify the geometrical similarity which of course includes the location and nature of the pressure connections and the geometry of the approach as well as the shape of the device itself. The necessity for exact similarity of outline and roughness can only be determined experimentally by comparing experiments with different values of these ratios. Restricting attention to systems which are geometrically similar even to the roughness, the equation reduces to

$$C_d = f\left(\frac{VD}{}, \frac{A_1}{A}\right)$$

One factor which is not generally considered in the equations for submerged discharge is that the average velocity head exceeds $V_1^2/2g$ by an amount which depends upon the velocity distribution in the approach pipe. The true average velocity head may be written as $\alpha_1 \dfrac{V_1^2}{2g}$ where $\alpha_1 \geqq 1$. With a parabolic velocity distribution, $\alpha_1 = 2$, while for a uniform distribution, $\alpha_1 = 1$. Badly distorted velocity distributions, such as are found downstream from an elbow or partly closed valve, may yield very large values of α. If the approach system is geometrically similar in two cases, α should have the same value at equal Reynolds numbers but deviations from strict similarity lead to unpredictable variations. Largely for this reason, it is desirable that the straight approach be sufficiently long that the velocity distribution be determined by friction alone. This length is of the order of 20 diameters but varies with the degree of initial disturbance. In addition to affecting the energy relationships, changes in velocity distribution also affect the contraction of jets issuing from orifices.

At a contraction, the velocity change tends to be the same for all elements of the liquid and at the section where the flow first

becomes parallel, the velocity distribution tends to become uniform and α approaches unity. The true average change in kinetic energy is approximately

$$\left.\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right)$$

and is greater than that assumed in deriving Eqs. (5.11) and (5.12). The coefficient obtained from these equations is greater than would be obtained if the true average velocity head were used.

Whirling components of flow, such as are caused by elbows, have an appreciable effect on the coefficient since the pressure is then not uniform over the cross section. The angular velocity increases as the fluid moves through the converging section toward the throat owing to the constancy of the moment of momentum and this causes the radial gradient of pressure at the throat to be greater than in the approach pipe, resulting in an underregistration of the differential head and an excessive coefficient.

The loss of head caused by differential meters varies greatly with the type. This loss is generally not the same as the differential head since even in the case of an orifice there is some recovery of velocity head as pressure. The loss should be measured between points well above and well below the meter and should be taken as the increment of loss over that which would occur by friction alone in the same length of pipe. In general, this loss may be expressed as $h_L = n\frac{V^2}{2g}$, where n is a function of the type of meter, the roughness, and the Reynolds number.

The fact that the hydraulic roughness of metal surfaces cannot be expressed quantitatively does not make impossible the correlation of data. In the first place, the effect of roughness may be eliminated completely by using the same meter with different fluids. Secondly, it is the relative roughness which is involved and one can assume that the same type of surface will have approximately the same absolute roughness in meters of different size, and that the ratio of the relative roughness of two meters of the same material and manufacture is inversely proportional to the diameters.

74. Venturi Meter.—Figure 82 shows a typical venturi-meter installation. The metering element is the converging portion

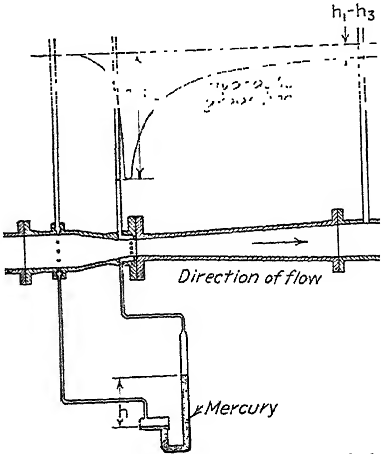

FIG. 82.—Typical venturi-meter installation. Adapted from "Fluid Meters Report," A.S.M.E.

between the main section and the throat. The diverging portion serves to convert the velocity head at the throat back into pressure head.

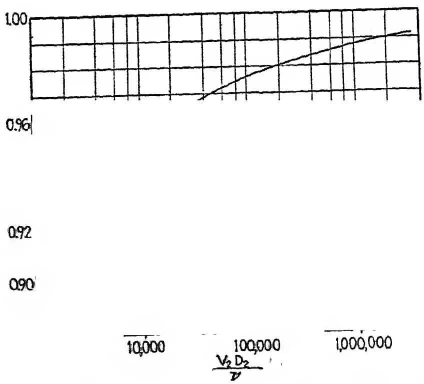

FIG. 83.—Coefficient of discharge for venturi meters. Ratio of inlet to throat diameter, 2 to 3.

Figure 83 shows the variation of the coefficient of discharge, C_d, of venturi meters for ratios of inlet to throat diameter between

2 and 3. The coefficient represents the effect of friction losses and velocity distribution only, since the contraction is fixed by the solid boundaries. The Reynolds number has been based upon the velocity and diameter at the throat.

The loss of head in a venturi meter is much less than the differential head. J. W. Ledoux found as a result of experiments with water on 24 tubes having diameter ratios between 1.5 and 3.0 that the total loss in feet could be expressed as

$$\overline{403}$$

or approximately as

$$h_L = 0.132\frac{V_2^2}{2g}$$

where V_2 is the velocity at the throat. The throat velocities in these experiments ranged from 1 to 40 ft. per sec. The absolute

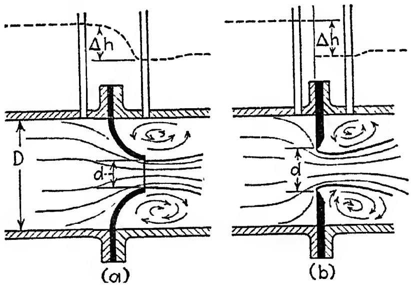

FIG. 84.

throat pressure in a venturi meter should be measured or computed to guard against cavitation.

75. Pipeline Nozzles and Orifices.—Figures 84*a* and 84*b* show the general arrangement of pipeline nozzles and orifices, respectively. The differential head is measured between points at the side wall above and below the diaphragm. The distribution of pressure at the pipe wall is shown *schematically* in the figures. The shape of the actual longitudinal pressure distribution curve varies with the area ratio and other factors.

Equation (5.11) may be used or computations may be further simplified by reducing this equation to

$$Q_a = \qquad (5.13)$$

Here

$$C = C_d\sqrt{\frac{A_1^2}{A_1^2 -}}$$

Equation (5.13) is in convenient form for computation and might just as well be used since the coefficient C_d depends for its value upon the area ratio in spite of the fact that the velocity of approach was included in the derivation of the equation.

It is evident that the value of the coefficient will vary with the location of the pressure taps. Taps near the orifice give a greater differential reading than at a distance and this arrangement appears to be preferable because it decreases the percentage error in this measurement. It also permits making the pressure

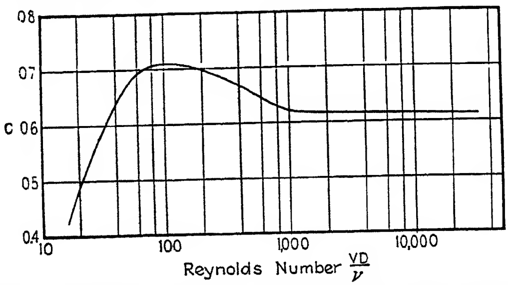

FIG. 85.—Typical coefficient of discharge for pipeline orifices of the International Standard type.

connections integral with the orifice or nozzle. Orifices used at the University of California have the pressure connections drilled through the plate itself. The International Standard orifices and nozzles are built with a slotted pressure connection against the orifice face and at the same radius as the pipe. Other types have the pressure connections at varying distances from the diaphragm.

Both C and C_d for nozzles and orifices may be expressed as functions of the diameter ratio, the Reynolds number, and the geometry of the design. Above a certain Reynolds number, the coefficient becomes constant, the critical value varying slightly with the diameter ratio. Computations of the Reynolds number may be based upon the diameter either of the pipe or of the aperture. The two values are related by the equation,

$$\text{Re}_{D_1} = \frac{V_1D_1}{\nu} = \frac{V_2D_2}{\nu} \cdot \frac{A_2}{A_1} \cdot \frac{D_1}{D_2} = \text{Re}_{D_2}\sqrt{m}$$

Here, V_2 is the average velocity in the plane of the opening of diameter D_2, and $m = (D_2/D_1)^2$. For the International Standard nozzles and orifices, the critical Reynolds numbers and the coefficients above these values are:

Area ratio m	Nozzles		Orifices	
	Critical Re_{D_1}	C	Critical Re_{D_1}	C
0.05	60,000	0.987	20,000	0.598
0.10	63,000	0.989	30,000	0.602
0.20	85,000	0.999	50,000	0.615
0.30	130,000	1.016	73,000	0.634
0.40	180,000	1.045	95,000	0.661
0.50	280,000	1.096	130,000	0.696
0.60	....	.....	150,000	0.742
0.70	....	.....	170,000	0.806

Figure 85 shows the general trend of the coefficient for pipeline orifices of the International Standard type.

Many other experiments on diaphragm orifices and nozzles are available. A good source of information on American industrial practice in metering is the report of the Fluid Meters Committee of the American Society of Mechanical Engineers. For precise work, direct calibration volumetrically or gravimetrically is recommended in place of application of formulas. If calibration is not possible, the coefficients should be used only in the formula from which they were computed, and the equipment should duplicate that for which the coefficients were obtained.

76. Short Tubes.—For certain services, what amounts to a short pipe is used for metering in place of an orifice or nozzle. This device is known as the *short tube* and is particularly applicable where the diameter of the opening is small in comparison with the thickness of the wall in which it is to be placed. Examples of such situations are the jets in carburetors, small nozzles in the concrete walls of water-treatment plants, and ordinary roadway culverts. Defining the coefficient as before,

$$= CA_2$$

and

$$= f\left(\frac{VD}{\nu}, \frac{L}{D}, \frac{k}{D}\right)$$

The principal dimensionless ratios are the Reynolds number and the length-diameter ratio. Figure 86 shows a typical curve of the coefficient as obtained by Zucrow.

77. Submerged Gates and Sluices.—If the downstream water level is well above the upper edge of gates, sluices, and other outlet structures, the flow conditions are essentially the same as for submerged orifices and nozzles and no special attention need be given them from the theoretical viewpoint. The coefficient of discharge varies with the design and, in general, must be determined by means of full-scale or model tests if essentially the same structures have not previously been tested.

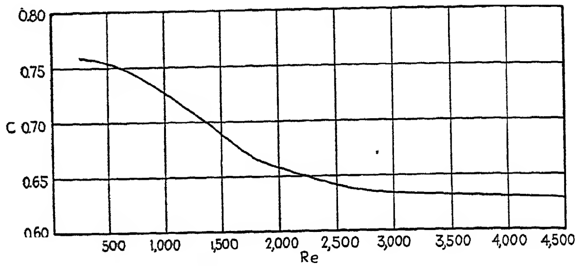

FIG. 86.—Coefficient of discharge for carburetor jets discharging benzol. (*After Zucrow.*)

78. Loss of Head.—It is to be noted that in all of the submerged flow devices mentioned except the venturi meter, the discharging stream shears into a large body of fluid and a shock loss may be expected to occur. In Chap. III, it was shown that the loss of head at a sudden increase of area is

$$h_L =$$

where V_2 is the initial velocity and V_3 the final velocity in a chamber of parallel sides. Inserting the continuity equation,

$$h_L = \frac{}{2g}$$

$$\frac{h_L}{h} = C^2\left(1 - \frac{A_2}{A_3}\right)^2$$

If the expansion of the jet is guided by solid boundaries which

diverge gradually, the loss can be greatly decreased. The optimum angle between conical sides is around 5 deg.

79. Effect of Pulsations.—The equations and coefficients mentioned in the preceding sections are said to apply only to steady flow even though this ideal condition never exists in practice. By exercising care, the pulsations in pressure and discharge can be made relatively small and presumably the published experimental coefficients have been obtained under reasonably steady conditions. The practical question then arises as to what is the effect of marked pulsations in flow on the discharge equations. Measurement of the flow of air to engines and compressors involves this problem. Before considering the magnitude of the effect it is well to remark that the sound procedure is to reduce the amplitude of the fluctuations, but this is not always possible.

In considering this problem it will be assumed that:

1. The instantaneous rate of flow may be computed from the steady-flow equations by using the instantaneous pressure difference. (This assumption appears to be justified by Gibson's experiments on the effect of waves on weir discharge.)

2. The manometer or other pressure measuring device is so damped and has a natural period such that it indicates the true time average of the instantaneous pressures.

The indicated average discharge, Q_e, computed from the average head, h_{av}, is

$$Q_e = CA \tag{5.14}$$

The instantaneous discharge, Q, will be assumed to be composed of an average value, Q_{av}, and a superimposed sinusoidal fluctuation. The time average of the fluctuating component is zero from the definition of Q_{av}. The equations follow:

$$\begin{aligned} Q^2 &= (Q_{av} + Q_0 \sin \omega t)^2 \\ &= C^2A^2(2gh) \end{aligned}$$

$$h_{av} = \frac{1}{2\pi}\int_0^{2\pi} h\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{(Q_{av} + Q_0 \sin \omega t)^2}{2gC^2A^2} dt$$

$$Q_{av}^2 + \frac{Q_0^2}{2}$$

From Eq. (5.14),

$$h_{av} = \frac{Q_e^2}{2gC^2A^2}$$

$$\frac{Q_{av}}{Q_e} = \frac{1}{\sqrt{1 + \frac{1}{2}\left(\frac{Q_0}{Q_{av}}\right)^2}}$$

The indicated average discharge Q_e exceeds the true average by an amount which depends upon Q_0, the semi-amplitude of the fluctuation in discharge. If $Q_0/Q_{av} = 0.1$, $Q_{av}/Q_e = 0.99$. If an additional indicating mechanism is provided which follows the head variations, Q_0 can be computed with sufficient accuracy to estimate the correction to be applied to Eq. (5.14). The correction is always in the same direction, the indicated rate of discharge being in excess of the true value. The ratio of the indicated to the true average discharge depends on the form of the fluctuations and a different result would have been obtained if a wave form other than sinusoidal had been assumed.

80. Effect of Compressibility.—For gases, the compressibility affects the flow both by changing the velocity and by altering the contraction, unless the contraction is guided by solid boundaries. If the pressure drop exceeds about 1 per cent of the absolute pressure, these effects become appreciable and modification is necessary, either of the basic equation or of the coefficient.

In Chap. III it was shown that the theoretical equation for the steady, adiabatic, frictionless flow of a perfect gas through any sort of convergent passage is

$$V_2 = \sqrt{\frac{2gn}{n-1}\frac{p_1}{w_1}\frac{1-(1-x)^{\frac{n-1}{n}}}{1-m^2(1-x)^{\frac{2}{n}}}}$$

where V_2 = the velocity at the contracted section.

n = the exponent in the adiabatic equation pv^n = const.

p_1 = the absolute upstream pressure.

w_1 = the upstream weight per unit volume.

$x = \dfrac{p_1 - p_2}{p_1}$.

$m = A_2/A_1$.

The theoretical weight rate of discharge is then

$$W = \qquad = A_2 w_1 (1 - \quad ^{1}$$

In the derivation of the adiabatic flow equation, the area A_2 was treated as being known in all instances whereas this is true only for nozzles, venturi meters, and short tubes. Just as was found to be true for orifices discharging liquids, gases experience a contraction of the jet on leaving a sharp-edged aperture and the stream is under the constant pressure p_2 only at the vena contracta. To simplify the equation, however, the velocity V_2 and the pressure p_2 are assumed to exist over the entire area at the least section A_2, and the corrective coefficient is made to include the effect of contraction as well as friction. This assumption is the same as was made to obtain the basic equation for the discharge of liquids through weirs and orifices. The equation representing the actual flow for all types of differential meters is then

$$W = C_1 A_2 w_1 \sqrt{\frac{2p_1}{\rho_1(1 - m^2)} \frac{n}{n - 1}\left[1 - (1 - x)^{\frac{n-1}{n}}\right]} \qquad (5.15)$$

where C_1 is an experimental coefficient which may be used in this equation only. The unwieldy form makes it desirable that, if possible, a simpler expression be adopted as the basic equation and the criterion of the desirability of making such simplifications is the possibility of representing the variations of the new coefficient in simple terms. In other words, the corrective coefficient in the simplified expression will be a function of the terms omitted from the precise equation as well as of the quantities determining C_1, and it is possible that the advantage will lie only in greater ease in writing down the equation and not in the application to actual problems.

To simplify Eq. (5.15) we shall assume that x is small and expand the term $(1 - x)^{\frac{n-1}{n}}$ by the binomial series, giving

$$\frac{n}{n - 1}\left[1 - (1 - x)^{\frac{n-1}{1}}\right] = x\left(1 + \frac{1}{2n}x \cdots\right)$$

For small values of x, $p_1 = p_2$ and $w_1 = w_2$ approximately,

Bean, Buckingham, and Murphy found that the coefficient C_2 could be represented for $Re_d > 200{,}000$ by

$$C_2 = 0.597 + 0.09m^2 - 0.115(x + x^2)(1 + 1.5m^2)$$

This equation applies to pressure taps at or near the orifice. Later work by Bean, Benesh, and Buckingham based the coefficient on the equation

$$W = C'A_2\sqrt{2gw_1(p_1 - p_2)}$$

which is the hydraulic formula. The coefficient was found to be represented by

$$\begin{aligned} C' &= CY \\ Y &= 1 - (0.40 + 0.46m^2)\frac{x}{n} \\ C &= 0.597 + 0.41m^2 \\ Re_d &> 200{,}000 \end{aligned}$$

For the International Standard orifices and nozzles, the equation defining the coefficient is taken to be

$$W = C\epsilon A_2\sqrt{2gw_1(p_1 - p_2)} \tag{5.17}$$

where C is the coefficient which would apply to liquids at the same Reynolds number [see Eq. (5.13)] and ϵ is a factor representing the expansion. The theoretical equation for ϵ may be obtained from Eqs. (5.15) and (5.17). Figure 87 shows experimental values of this expansion factor for International Standard nozzles and orifices. The corresponding values of C are those stated previously. These coefficients are applicable above the critical Reynolds number at which the coefficients become constant.

Flow at supersonic velocities, as in the expanding nozzles of steam turbines, is considered briefly in Chap. VIII.

SEMISUBMERGED DISCHARGE

The preceding sections have considered structures or devices in which the surface of the discharging jet was wholly in contact, either with a large mass of the same fluid, or with a fluid of negligible density. There are, however, many situations falling in neither of these classifications. The discharge over a broad-crested weir is a typical example for here the surface is under

atmospheric pressure but the interior elements of the fluid are under pressures which vary with velocity, curvature, and depth. Gates and bottom outlets frequently operate with partial submergence. In some instances, the flow conditions may be analyzed from considerations of energy or momentum but, in general, the wetted surface is sufficiently irregular that experimental results are necessary in computing the rate of discharge or the water-surface curve.

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Problems

1. A sharp-crested 90-deg. triangular weir discharges 0.31 cu. ft. per sec. with a head of 0.42 ft. Neglecting velocity of approach, what is the discharge coefficient? *Ans.* 0.635.

2. If a small rectangular weir discharges 0.11 cu. ft. per sec. for a head of 0.10 ft. and 0.82 cu. ft. per sec. for a head of 0.40 ft., determine K and n in the equation $Q = KH^n$. *Ans.* $K = 3.1$, $n = 1.45$.

3. The following experimental data were obtained using a 24-in. rectangular weir:

Discharge, Q, cu. ft./sec.	Head above crest, H, ft.
0.274	0.117
0.758	0.235
1.95	0.448
4.03	0.735
7.78	1.155

(*a*) Compute and plot on graph paper the coefficients of discharge as a function of the head above the crest. (*b*) Find values of K and n in a formula of the type $Q = KH^n$.

4. Compute actual discharge of water from a large rectangular vertical orifice (12 × 12 in.) under 4-ft. head above the center of the orifice. Compare the discharge with that obtained from formula for small orifices.

5. A sharp-edged orifice 1 in. in diameter discharges horizontally from the side of a tank against a large vertical plate which is perpendicular to the jet. The measured discharge is 0.0542 cu. ft. per sec. and the force exerted on the plate is 1.65 lb. The head on the orifice is 4 ft. Compute (*a*) discharge coefficient of the orifice, (*b*) velocity coefficient of the orifice, (*c*) contraction coefficient of the orifice.

6. Water issues from a 2.2 inch circular orifice in a vertical plane under a head of 16 ft. The rate of discharge is 0.55 cu. ft. per sec. If the coefficient of velocity is 0.96, find (*a*) coefficient of discharge, (*b*) coefficient of contraction, (*c*) the diameter of the jet.

7. A square tank with 4-ft. sides has a fully streamlined nozzle in the bottom. Compute the time required for the level to fall from 6 to 2 ft. above the discharge plane of the nozzle when the nozzle diameter is 3.0 in. and the discharge coefficient is constant at 0.97. *Ans.* 87.1 sec.

8. Compute the exact theoretical discharge through a vertical 1-ft. square orifice in the side of a tank if the free liquid surface in the tank is 6 in. above the top edge of the orifice. Compare with the result obtained by multiplying the theoretical velocity at the center by the orifice area.
Ans. 7.95 cu. ft. per sec., 8.025 cu. ft. per sec.

9. If the orifice of Prob. 8 is in a vertical partition between two tanks whose free liquid surfaces are 4 and 3 ft., respectively, above the top edge of the orifice, what is the theoretical discharge?
Ans. 8.025 cu. ft. per sec.

10. A venturi meter having a 6-in. throat and 12-in. inlet diameter is installed in a vertical position with connections to a differential mercury manometer. What is the rate of flow if the deflection of the mercury column is 18 in. (sp. gr. of mercury 13.6) and the discharge coefficient of the meter is 0.98?

11. A circular 4-in. sharp-edged orifice is placed in a 6-in. water line. A water-mercury differential manometer connected across the orifice shows a deflection of 1.69 ft. What are (*a*) the theoretical discharge through the orifice, and (*b*) the discharge coefficient if the actual rate of flow is 2.22 cu. ft. per sec.? *Ans.* (*a*) 3.60 cu. ft. per sec. (*b*) 0.616.

12. Compute the Reynolds number for Prob. 11 and compare the discharge coefficient with the curves of Fig. 79. (Assume water temperature at 68 deg. Fahr.) *Ans.* Re = 848,000.

13. An air compressor is to be tested by means of a 2-in. nozzle with a discharge coefficient of 0.98 placed in the side of a large air receiver. The nozzle discharges into the atmosphere. The pressure in the receiver is 8 in. of mercury and the temperature is 80 deg. Fahr. The barometric pressure is 30 in. of mercury. Compute (*a*) the number of pounds of air discharged per second, (*b*) the number of cubic feet of air per second passing the tip of the nozzle. Compare results given by eqs. (5.15) and (5.16).

14. Determine the rate of flow of dry air in pounds per second discharged through a 2-in. by 1-in. venturi meter if the upstream and throat pressures are -2 and -7 in. of mercury, respectively, and the upstream temperature is 60 deg. Fahr. Assume adiabatic flow, atmospheric pressure at 30 in. of mercury, and that the discharge coefficient may be obtained from Fig. 83.
Ans. 0.203 lb. per sec.

CHAPTER VI

FORCES EXERTED BY FLUIDS

The rapid development of both the theoretical and practical phases of aviation has provided a large mass of data on the forces which fluids exert on certain types of objects and has also emphasized the basic principles applicable to all objects. Less spectacular but nevertheless important work in the design of ships has also contributed to our knowledge of the subject. By means of dimensionless criteria such as the Reynolds number, these data are made applicable to other fluids and it is the purpose of this chapter to review briefly some of the experimental results.

81. Origin of Resistance.—The force exerted on an object by a fluid can always be obtained by integrating the components of the normal and tangential forces over the entire surface of the object. This statement applies to all objects whether wholly or partially submerged but it is evidently difficult to apply. In practical applications it is generally over-all effects, such as the drag, lift, and moment, which are of importance and in the study of these quantities it is essential that a distinction be drawn between objects which are submerged and those which are at or near the surface of separation of two fluids. These two situations must be treated separately because the forces exerted may be materially altered by the formation of surface waves. To be more specific, the resistance of a ship or a bridge pier is not one-half the resistance of a similar submerged object symmetrical above and below the water line because the system of surface waves generated carries away a certain amount of energy. This transfer of energy is associated with an increase in the force exerted, over and above the force on the same object when wholly submerged.

Normally, objects immersed in gases may be considered as *submerged* since a gas expands to fill any space in which it is placed. However, it should be noted that waves may form at

the surface of separation of gases of different densities. A similar situation may exist at the surface of separation of different liquids or of the same liquid at different temperatures or with different concentrations of dissolved or suspended material. The most conspicuous example of the formation of waves is at the surface of separation of a liquid and a gas and it is this problem which has received most attention. For the purpose of classification, bodies will be considered as *submerged* when they are sufficiently far from a surface of separation that the energy of wave formation is small as compared with that dissipated in skin friction and eddies. This definition has a dynamical aspect in that for a given depth below the free surface of a liquid a body may be regarded as wholly submerged in this sense only below a certain velocity. Above this critical velocity the conditions of true submerged flow may be destroyed either by the generation of appreciable waves at the surface or by the formation of cavities in the interior of the liquid.

Another important problem is the force exerted by a free jet on an object such as the bucket of an impulse turbine. The essential features of the solution of this type of problem were treated in Chap. III in connection with the momentum equations and will not be included here.

SUBMERGED OBJECTS

82. Force Coefficients.—In Chap. IV, application of the principle of dimensional analysis showed that the force exerted by a fluid stream on an object submerged in it could be represented as

$$F = CA\tfrac{1}{2}\rho V^2$$

The factor $\frac{1}{2}$ has been inserted to conform to American practice in computing aerodynamic coefficients. Here, C is a dimensionless coefficient which can in general be represented by

$$C = f\left(\frac{VL}{\nu}, \frac{k}{L}, \frac{l_1}{L}, \frac{l_2}{L}, \cdots, \frac{l_n}{L}, \alpha_1, \alpha_2, \cdots, \alpha_n\right) \tag{6.1}$$

where k/L represents the relative roughness of the object, l_1/L, l_2/L, . . . , l_n/L are the ratios representing the geometry of the object and *of the flow system,* and α_1, α_2, . . . , α_n are angles representing the setting of the object or objects relative to the

fluid stream. For example, in the measurement of the lift and drag characteristics of model airfoils, the similarity should include not only the shape and angle of attack of the airfoil but also the geometry of the wind tunnel, which affects the turbulence in the air stream. The velocity V is generally the velocity at a distance relative to the object. The area A is any convenient area and may be chosen arbitrarily since geometrical similarity is specified in Eq. (6.1). The extent to which the geometrical similarity of the flow system and the object must be realized in practice for equality of the coefficient can only be determined experimentally.

The power expended in overcoming fluid resistance is F_vV, where F_v is the force in the direction of the relative motion. This amount of power must be supplied from an external source to drive the object through a stationary fluid at velocity V or must be supplied by the fluid if the object is stationary. The momentum relationship also may be applied in the form

$$F_s = \int_0^w \frac{\Delta V_s}{g} dw$$

where F_s is the force and ΔV_s is the change in the velocity in the S-direction. The integration is to be carried out over a control surface surrounding the object, as explained in Chap. III. If the flow is symmetrical about an axis parallel to V, there will be no net transverse force. If the flow is not symmetrical about an axis parallel to V, there will generally be a transverse force and a corresponding deflection of the fluid stream.

For convenience, the force exerted may be regarded as the resultant of the force caused by skin friction along the surface, the force related to the generation of eddies, and the force necessary to cause any additional change in momentum. These forces are not independent since the skin friction affects the generation of large eddies and these two effects influence the change in momentum but, nevertheless, some simplification of the problem is brought about by considering the relative importance of these forces in specific cases. For example, if a thin flat plate is placed parallel to a turbulent stream of fluid, the force exerted is due to skin friction, and the change in momentum of the air stream as well as the force on the plate is due to this friction force alone. If the same plate is turned so as to be perpendicular to the direction of the fluid, the skin friction

becomes secondary and the main cause of the resistance is the generation of large eddies at the edges of the plate. Energy of directed motion is changed into energy of rotation by the application of a force which is, of course, related to the change in momentum of the stream. Setting the plate diagonally with the direction of flow will deflect the stream and give rise to a transverse or *lift* force. The relative importance of skin friction and of eddy formation depends on the angle at which the plate is set.

In Chap. IV, it was shown that the friction coefficient in pipes approaches a constant value as the Reynolds number increases and that the Reynolds number at which this coefficient becomes constant decreases with increasing roughness. Similarly, in Chap. V, it was found that above a certain Reynolds number the coefficients of orifices, nozzles, and other flow devices ultimately become constant. The Reynolds number itself is the ratio of the inertia to the viscous forces and it appears reasonable to conclude that the dimensionless coefficients tend to become constant whenever inertia effects predominate. Consequently, one would expect the force coefficients of objects to become constant at lower Reynolds numbers if the objects cause abrupt changes in velocity or high turbulence than if they are gradually tapered and smooth.

83. Stagnation Point.—Whenever the fluid passes completely around an object, there is a point near the forward end at which the flow divides and the pressure rises above that at a distance by an amount equal to $\frac{1}{2}\rho V^2$, where V is the velocity at a distance. This phenomenon was discussed in Chap. III in connection with the pitot tube. The pressure rise is known as the *dynamic pressure*, and the point at which it occurs is the *stagnation point.* The force coefficients used in this chapter are so defined as to equal the ratio of the actual force to the product of the dynamic pressure and the area.

84. Skin Friction.—The frictional resistance of flat plates placed parallel to the direction of motion of the fluid was treated theoretically in Chap. IV as part of the discussion of turbulent flow. However, the empirical and theoretical equations, which may be used to compute frictional forces, were not included there since they are more directly applicable to the material which follows.

The experiments of William Froude were concerned with the frictional resistance of flat plates towed through water, and formed a part of his researches on ship resistance. Accordingly, the velocities and sizes which he tested covered the range of such experiments and the constants are applicable to fresh water at normal temperatures. Later experimental data are preferable for computations, but his results illustrate the general trend. Adopting as the basic equation $R = fAV^n$, the coefficients are as follows:

Surface	Length								
	2 ft.			8 ft.			50 ft.		
		f*	f_x*	n	f*	f_x	n	f*	f_x
Paraffin.	1.95	4.2	4.1	1.94	3.6	3.0			
Varnish.	2.00	5	3.9	1.85	4.6	3.7	1.83	3.7	3.3
Sand....	2.00	9.0	7.3	2.00	6.2	4.9	2.00	4.9	4.6

* Values of f and f_x have been multiplied by 10^3.

The coefficient f gives the average resistance in pounds per square foot over the length specified, measured from the leading edge, while f_x gives the local resistance in pounds per square foot at the location specified. The exponent n is 2 for the rough surface but is slightly less for smooth surfaces. The resistance per unit area decreases with distance from the forward edge, *i.e.*, for a given velocity V at a distance, the contribution of each additional unit of area towards the total resistance decreases as the length is increased.

On the basis of experiments in smooth pipes, Prandtl and von Kármán obtained the equations

$$\frac{R}{\frac{1}{2}\rho V^2 bx} = C_f = 0.074\left(\text{[illegible]} \right. \tag{6.2}$$

$$\frac{\tau_0}{\text{[illegible]}} = c_f = 0.059\left(\text{[illegible]} \right.$$

Here, C_f is the coefficient of the average resistance up to the point x measured from the leading edge and c_f is the coefficient of local friction at this position. Both coefficients are dimension-

less. Equations (6.2) were obtained on the assumption of a certain exponential formula for the friction in pipes and are valid for smooth plates only to

A later theoretical analysis of the problem by von Kármán showed that the general equation for smooth plates should contain only two numerical constants, one of which is κ, the universal constant of the turbulence theory. Measurements by Kempf on the forces on small movable plates placed at different points

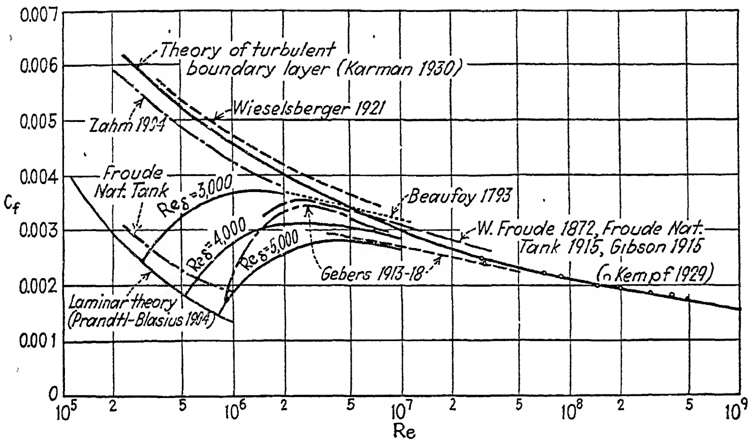


FIG. 88.—Skin friction of smooth plates compared with the theory. (von Kármán in *Journal of Aeronautical Sciences*, January, 1934.)

along the surface give the values of these constants as in the following equation:

$$\frac{1}{C_f^{\frac{1}{2}}} = 1.7 + 4.15 \log_{10} (\text{Re}_x C_f) \qquad (6.3)$$

The constant 4.15 corresponds to $\kappa = 0.395$. Figure 88 is a comparison of Eq. (6.3) with various experimental results.

Figure 88 also shows that there is a transition from laminar to turbulent flow at a fairly definite value of the Reynolds number just as in the case of smooth pipes. The exact transition point depends upon the turbulence of the fluid approaching the plate. Expressed in terms of a Reynolds number based on the local

thickness of the boundary layer ($Re_{z0} = Vz_0/\nu$), von states that the critical value lies in the range $1600 < Re_{z0} < 6000$ (see Chap. IV, page 133 for computation of z_0).

The effect of roughness on the resistance of flat plates may be treated in the same manner as in the case of pipes. The basic difficulty is to express the roughness of common surfaces in quantitative terms and one must still rely on empirical data for the friction coefficients of rough surfaces.

85. Vortex Formation—von Kármán Vortex Street.—Referring to Fig. 89, which shows the flow around the middle portion of a plate, long in the direction perpendicular to the sketch, the lines AA and BB separate the main flow from the dead fluid behind the plate. It is known from experiment that at a distance

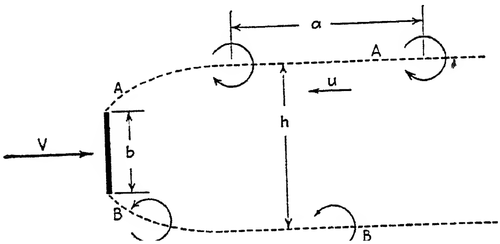


FIG. 89.

behind the plate these surfaces of discontinuity are marked by a staggered series of vortices in parallel rows with the sense of rotation in each row as shown by the arrows. von Kármán investigated the arrangement of the vortices and found that the staggered arrangement is stable when $h = 0.281a$. If the velocity of the vortices *relative* to the fluid is u, the frequency of vortex formation at one edge of the plate is $f = V - u$ and the energy stored in the vortex system requires that the drag on the plate be

$$D = \frac{1}{2}\rho b V^2 \left(\frac{h}{b}\right)\left[5.66\left(\frac{u}{V}\right) - 2.24\left(\frac{u}{V}\right)^2\right]$$

or

$$C_D = \frac{h}{b}\left[5.66\frac{u}{V} - 2.24\left(\frac{u}{V}\right)^2\right] \qquad (6.5)$$

The theory is incomplete in that it does not permit prediction of h/b and u/V for bodies of different shapes, but if these quantities are measured, the values of D or C_D computed from Eqs. (6.4) and (6.5) are in good agreement with experiment.

In most problems met in engineering practice, the flow is not even approximately two-dimensional and vortices form all around the object. Whether these vortices form as a continuous spiral or a series of torus vortices is not definitely established. However, the loss of available energy is directly related to the strength and frequency of vortex formation which are in turn dependent upon the shape of the body and the velocity. For this reason the force associated with vortex formation is called the *form drag* to distinguish it from the tangential forces due to skin friction. Angular shapes give rise to large form drag while for well-faired shapes, such as airfoils, it is relatively small under usual conditions. Even airfoils generate vortices due to their finite length and give rise to what is known as the *induced drag* (see Sec. 89). Experiments by Carter show that projections from a solid boundary, such as those forming the roughness of a pipe, generate vortices which break away and move into the interior of the fluid along spiral paths. The formation of these vortices is probably the principal cause of frictional resistance in turbulent flow.

Although the time average of the transverse forces on a body symmetrical about an axis parallel with the direction of flow is zero, the periodic formation of vortices in a staggered series must give rise to instantaneous transverse forces of a periodic nature. These periodic forces will tend to set the body in transverse vibration and a condition of resonance will arise if the period of vortex formation coincides with one of the natural periods of the body or its supports. The Aeolian harp, which "sings" in the wind, and the motion of a branch in a stream of water are examples of such resonance. The swaying of electric cables and the "flutter" of airplane wings and control surfaces result from the same cause.

86. Interference.—Before discussing the experimental data on the forces exerted on different objects, it is well to mention that the forces are altered by the presence of other objects. In some portions of the fluid field surrounding an object, the velocities are increased over those which would exist at the

same point without an obstruction while at other points they are decreased. The force on a second object may therefore be greater or less than on the same object when tested alone, depending on its position and on the character of the velocity field. The second object alters the forces on the first and so on. Certain types of interference problems have been investigated in great detail, such as, for example, the mutual interference of

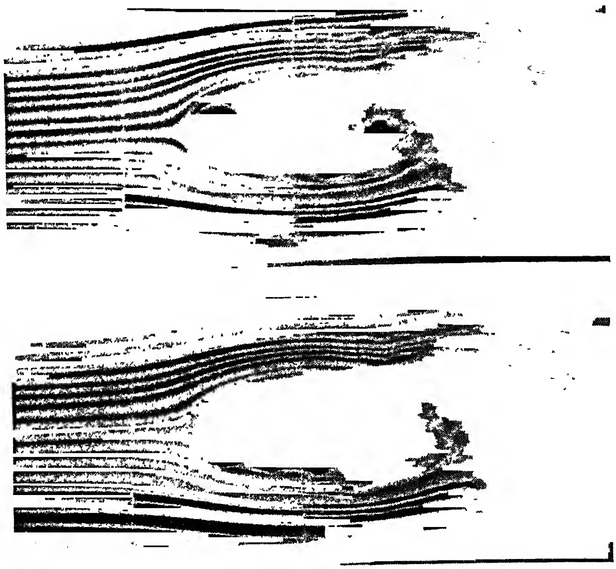


FIG. 90.—Smoke photograph of flow around a stationary cylinder. Air velocity, 7 ft. per sec. Note the row of staggered vortices in the wake. (*Courtesy National Advisory Committee for Aeronautics.*)

the wings of a biplane, but in general the effect can only be estimated approximately, if at all. The force coefficients quoted in the following sections are approximately those which would apply if the object were located at a considerable distance from other objects and from any solid boundary.

87. Symmetrical Flow.—If a symmetrical object is placed with its axis of symmetry parallel to the direction of the relative flow, the stagnation point occurs at the upstream end of this axis. The main stream of fluid follows the surface for a distance which depends on the curvature. If the object is well streamlined, the

flow may follow the surface to the downstream end but, in general, the main stream breaks away leaving the downstream portion of the object in contact with fluid which does not partake of the general motion. The stream leaving the surface exerts a "suction" effect which reduces the pressure over the downstream surface below the pressure at the boundary of the main flow and may give rise to dynamic forces in excess of the projected area times the dynamic pressure at the stagnation point. Figure 90 shows the flow around a stationary cylinder.

1. *Flat Plates.*—For flat plates placed normal to a turbulent stream of fluid, the resistance is approximately constant. The variation with Reynolds number has not been studied extensively, but Dryden states that it is probably constant for Re > 1000. The same author states that for circular disks the average pressure rise on the upstream face is $0.78(\frac{1}{2}\rho V^2)$, while the pressure reduction on the downstream face below the static pressure at a distance is constant and equal to about $0.34(\frac{1}{2}\rho V^2)$. A few measurements of the coefficient C in Eq. (6.1) follow:

Authority	C	Size of plate
Cavonetti....	1.12	36 in. diam., circular
Dryden[1].....	1.15	Circular
Eiffel........	1.10	10 in. sq.
Dines........	1.12	12 in. sq.
Eiffel........	1 12	14 in. sq.
Eiffel........	1.18	20 in. sq.
Eiffel........	1.22	27 in. sq.
Eiffel........	1.24	39 in. sq.
Stanton......	1.04	24 in. sq.
Stanton......	1.24	60 in. sq.
Stanton......	1.24	120 in. sq.
Dryden[1].....	1.12 ± 3%	Square
Prandtl[1].....	1.10	Rectangular, ratio of sides 1:1
Prandtl[1].....	1.19	Rectangular, ratio of sides 1:4
Prandtl[1].....	1.29	Rectangular, ratio of sides 1:10
Prandtl[1].....	1.45	Rectangular, ratio of sides 1:20
Wieselberger[1]	2.0	Infinite strip normal to flow

[1] Average values for all sizes.

The effect of the length-width ratio (aspect ratio) is considerable owing to the increase in flow between the two sides as the ratio of peripheral length to area increases. The variation with

absolute size indicates either that the Reynolds number had some effect or that the interference of side walls or other details of the experimental arrangement was not negligible. The resistance of circular disks as a function of Re appears in Fig. 91. In recent years the resistance of flat plates has not received much attention because interest has centered on shapes more suitable for airplane structures.

2. *Circular Cylinders.*—The flow around infinitely long circular cylinders exhibits several peculiarities which are of interest. The coefficient in Eq. (6.1) is shown as a function of

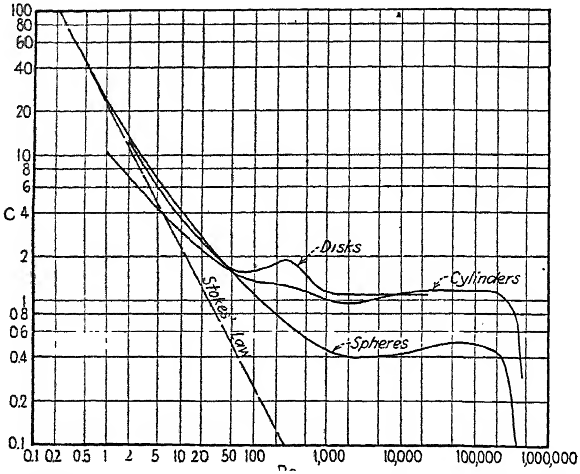


FIG. 91.—Coefficient of resistance for spheres, disks, and cylinders as a function of Reynolds number.

Reynolds number in Fig. 91. Above Re = 100, eddies begin to form periodically and for a narrow range of Re, the coefficient is approximately constant. It then drops to a minimum of approximately 0.85 at Re = 1800 and rises to a constant value of 1.2 for Re = 32,000. The coefficient then drops abruptly to 0.3 at a certain Reynolds number which depends on the turbulence in the air stream. The changes in the trend of the coefficients result from variations in the nature of the flow and illustrate very nicely the danger of extrapolating experimental data beyond the range of the experiments.

3. *Spheres.*—The drag coefficient of spheres, when plotted as a function of Reynolds number, shows the same abrupt decrease

at a value of the Reynolds number which depends upon the turbulence of the air stream. In fact, this phenomenon has been investigated so extensively that the turbulence of a wind tunnel may be expressed in terms of the Reynolds number at which the coefficient decreases abruptly. Figure 91 shows the variation in the drag coefficient as a function of Re, while Fig. 92 shows the effect of turbulence on the point at which the coefficient "breaks."

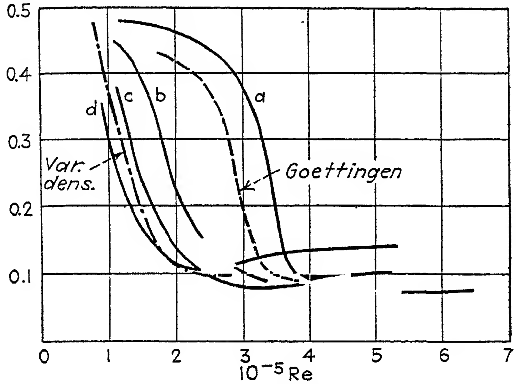
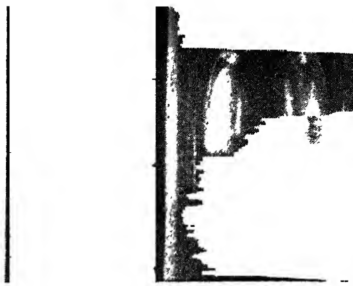


FIG. 92.—Sphere-drag coefficient C_D, versus Reynolds number Re, for various rind streams. Curves a, b, c, and d refer to the wind tunnel of the Guggenheim ıeronautics Laboratory of the California Institute of Technology, with no grid nd with grid 48 in., 20½ in., and 10½ in. upstream from the model, respectively. 'he curves for the N.A.C.A. variable-density and the Göttingen wind tunnels re also included. (*Millikan and von Kármán, Trans. A.S.M.E., March*, 1934, ı. 156.)

88. Settling Velocity.—In a number of engineering and cientific fields, the velocity with which solid particles, drops, or ›ubbles will rise or fall through a fluid is of importance. Trans-›ortation of pollen and dust by the atmosphere and of silt by treams, the determination of the size of fine materials, and the ›peration of air-lift pumps are examples. The motion of drops ınd bubbles is complicated by the fact that their shape and ıence their resistance to relative motion depends upon their ize and interfacial tension. Figure 93 shows an air bubble ising through oil while Fig. 94 shows the variation in the coefficient of resistance of air bubbles as a function of the Reynolds number. In order to emphasize the basic relationships, the following discussion will be directed chiefly towards the motion of solid particles.

When a solid particle is released in a fluid, it will fall or rise depending on whether it is heavier or lighter than the fluid and will approach a terminal velocity asymptotically. The time



$r_e = 0.0408$ ft. $V = 0.55$ ft. per sec.

FIG. 93.—Air bubble rising through livestock oil. r_e = radius of sphere of equal volume.

interval during which its velocity differs appreciably from the terminal value depends upon the difference in density and the resistance coefficient. At the terminal velocity, the fluid

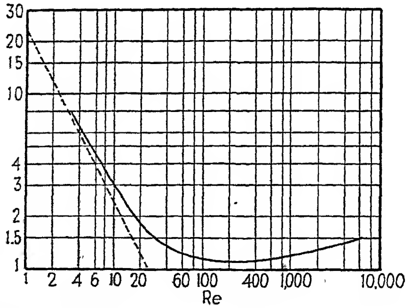


FIG. 94.—Coefficient of resistance of air bubbles as a function of Reynolds number.

resistance must equal the weight of the particle minus the force of buoyancy or

$$gv(\rho_2 - \rho_1) = C_R \tfrac{1}{2}\rho_1 A V^2$$

Here, v is the volume, ρ_2 the density of the particle, and ρ_1 that of the fluid, C_R is the drag coefficient, A is the projected area, and V is the velocity relative to the fluid. The value of the coefficient

evidently depends upon the shape of the particle and the Reynolds number.

Stokes investigated theoretically the velocity of spheres at low Reynolds numbers and obtained the equation

$$\frac{}{\text{Re}}, \qquad \text{Re} = \frac{2rV\rho_1}{}$$

where r is the radius of the sphere and μ_1 is the viscosity of the fluid. Substituting for C_R and Re and solving for the velocity,

$$\frac{2}{9} \qquad - \rho_1) \tag{6.6}$$

Equation (6.6) is known as *Stokes' law*. Experiments show that it is valid only up to Re = 0.5. Above this Reynolds number the actual fall velocity of spheres is less than that given by Eq. (6.6). To fix in mind the limit of applicability of Stokes' law, consider a sphere of specific gravity 2.65 falling through water at 70 deg. Fahr. The radius of such a particle corresponding to Re = 0.5 is 1.33×10^{-4} ft. or 0.0016 in.

The effect of shape on fall velocity may be surmised from experiments on the resistance of stationary objects. It is known that a flat object of uniform density tends to fall with its greatest projected area perpendicular to the direction of relative motion, sometimes with spiraling or tipping motion. Accordingly, if one compares particles of equal volume and specific gravity, the fall velocity will be greatest for spherical objects. If one compares flat objects, the fall velocity will decrease as the projected area departs from the circular form. Experiments on the fall velocity of quartz sand grains in water at 75 deg. Fahr. showed that the relationship between the diameter as given by a standard sieve analysis and the fall velocity could be represented by the equation

$$D = 0.094V^{1.109}$$

Here, D is in inches and V is in feet per second.

The density or specific weight of a mixture of solids and fluid is equal to the summation of the weights of solid and fluid present per unit volume only when the solids are falling with their terminal velocity. If the terminal velocity has not been attained, the effective weight per unit volume is equal to the

weight of an equal volume of fluid plus the summation of the downward force of resistance exerted on the fluid by the particles. This definition may be of some importance in the treatment of such problems as the transportation of mixtures in turbulent flow, where the fluid velocity is constantly changing.

The settling velocity is frequently used as a method of determining the size of finely divided material. Its limitations in this respect are evident from the effect of shape and its suitability depends upon the use which is to be made of the results. In flow problems, such as the transportation of solids by water and percolation through granular materials, the fall velocity itself is probably the most significant single characteristic of the material since it represents the combined effect of the size, shape, and density of the material and of the density and viscosity of the fluid.

89. Unsymmetrical Flow.—If a symmetrical object is placed with its axis of symmetry at an angle with the direction of the relative flow, a transverse force is exerted. Likewise, a transverse force generally acts on unsymmetrical objects however placed. Ship rudders and airplane control surfaces when in a turning position, guide vanes in pumps and turbines, and kites and airfoils are examples of objects which produce unsymmetrical flow systems and which depend for their effectiveness on the development of forces transverse to the direction of the relative flow. Of this class of objects, airfoils have been investigated most thoroughly and will be used to illustrate the general principles. In Fig. 95 the object is symmetrical but the flow is asymmetrical because the cylinder is rotating.

The lifting force on an airplane arises from the fact that the wings are inclined at an angle to the relative motion of the air, which is given a downward component of momentum. The magnitude of the force will depend upon the angle of inclination (known as the *angle of attack*), the Reynolds number, the relative velocity and density of the air, and the area and length of the wing. The force perpendicular to the relative motion at a distance is known as the *lift*, while that in the same direction is called the *drag*.

The conditions for dynamical similarity are the same as for other submerged objects, namely, that if two airfoils are geometrically similar, equality of force coefficients should exist at

equal Reynolds numbers. Most of the data on the characteristics of airfoils have been obtained in atmospheric wind tunnels. Consequently, the kinematic viscosity is very nearly the same for the model airfoil and the full-scale wing and the condition for similarity reduces to

$$VL = vl \tag{6.7}$$

For example, if the velocity of flight is 200 m.p.h., a one-tenth scale model airfoil should be tested at 2000 m.p.h., an obviously

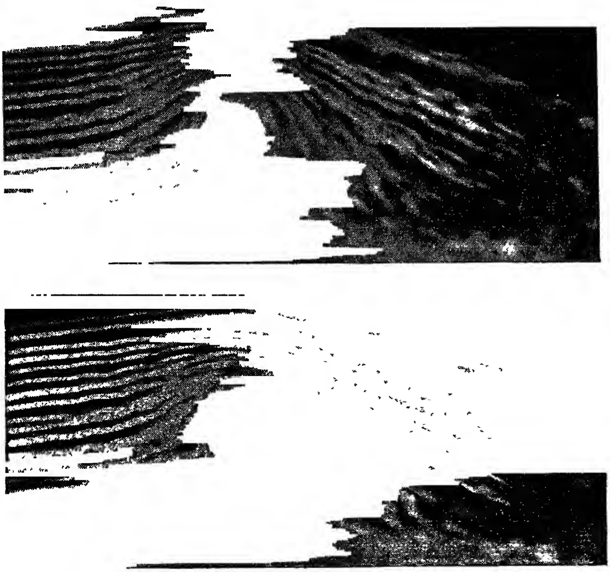


FIG. 95.—Smoke photograph of flow around a rotating cylinder. Air velocity, 7 ft. per sec. (*Courtesy National Advisory Committee for Aeronautics.*)

unsound requirement because this velocity is greater than that of sound and would result in a major change in the flow. Solution of this problem of model testing lies in the fact that the force coefficients become constant at relatively low Reynolds numbers and above this point Eq. (6.7) need not be satisfied.

The lifting force on an airfoil is the vector summation of the pressure forces on the surface less the component of tangential force. Consequently, a pressure difference must exist between the upper and lower surfaces with the greater pressure below. If the airfoil is very long or is placed between two parallel walls,

this pressure difference would be constant but if the length, or span, is finite, the pressure difference must decrease to zero at each tip, resulting in a tendency towards flow around the tip from the lower to the upper side. Thus in addition to downward deflection of the air stream, there is generated a spiral vortex system with a magnitude depending on the aspect ratio or ratio of span to chord. Generation of such a vortex system requires the expenditure of energy and has the effect of an added resistance which is known as the *induced drag*.

The forces on an airfoil of finite span are as follows:

1. A lift perpendicular to the direction of relative motion:

2. A drag in the direction of relative motion:

$$D = C_D \tfrac{1}{2}\rho A V^2.$$

The drag may be broken down into the profile drag, or the drag of the same type of airfoil but with infinite span, and the induced drag.

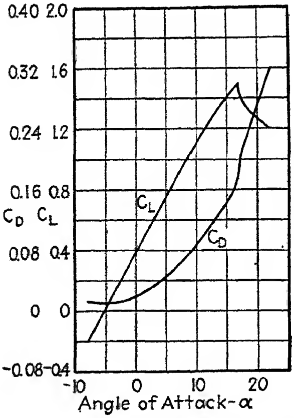


FIG. 96.—Typical lift and drag coefficients. Clark Y airfoil. Reynolds number 3×10^6.

Figure 96 shows typical experimental data on an airfoil having an aspect ratio of 6. Figure 97 shows separation from an airfoil above the point of maximum lift.

The method of correcting for the effect of aspect ratio is as follows: For elliptical loading, the coefficient of induced drag is related to the coefficient of lift by the equation

$$C_{Di} = \frac{C_L{}^2}{\pi R}$$

where R is the aspect ratio. The profile drag is obtained from the test data as

$$C_{Dp} = C_D - C_{Di}$$

The coefficient C_{Dp} remains the same for all aspect ratios and C_D can therefore be computed for other values of R than that used in the experiment.

The induced drag is accompanied by a reduction of the effective angle of attack amounting to

$$\alpha_i = \frac{C_L}{\pi R} \text{ (radians)}$$

The equivalent angle of attack of an airfoil of infinite aspect ratio is therefore

$$\alpha_0 = \alpha - \alpha_i$$

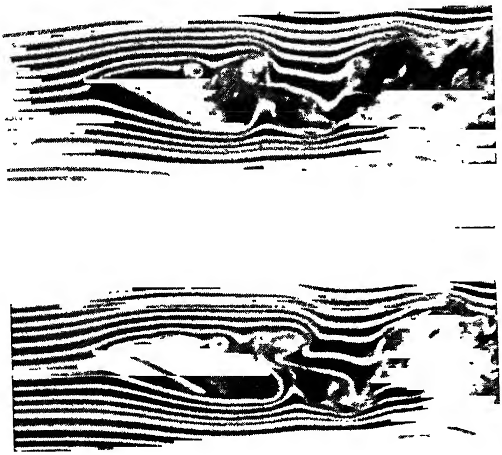


FIG. 97.—Smoke photograph of flow around a symmetrical airfoil section. N.A.C.A. 0012. (*Courtesy National Advisory Committee for Aeronautics.*)

where α_0 is the angle of attack for equal lift with an infinite aspect ratio and α is the geometrical angle. Test data obtained at one aspect ratio may be used to compute the characteristics at another aspect ratio in this manner.

Several features of the characteristics of airfoils should be noted. The lift coefficient reaches a maximum value at an angle of attack called the *burble point,* which recent experiments show is dependent on the turbulence of the air stream as well as the shape of the airfoil. The ratio of lift to drag reaches its maximum value at a much smaller angle, and over the useful range the lift coefficient is approximately a linear function of the angle of attack.

90. Wind Pressures.—Buildings and similar structures are submerged in air and the general principles involved in the prediction of pressures and forces caused by wind are the same as for the submerged objects previously mentioned. However, there are a number of differences among which are the following:

1. Only the maximum pressures, forces, and moments are important.
2. The wind may blow from any direction.
3. The exterior of a building is a combination of flat or curved surfaces which is seldom reproduced exactly in another structure.
4. The wind velocity varies with elevation and the distribution depends not only upon the meteorological conditions but upon the roughness of the surrounding area.

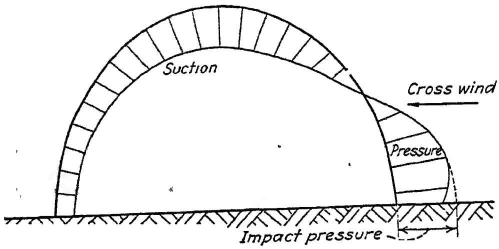


FIG. 98.

5. The wind intensity may be reduced or increased by the presence of adjacent structures.

Recalling the variations which occur in the force coefficients of flat plates, spheres, and other objects when measured under laboratory conditions, the impossibility of precise prediction of the forces on buildings is evident. However, the fact that the structural design aims at a strength which will withstand the maximum wind intensity makes it possible to establish arbitrary rules for design. A usual procedure is to provide for a horizontal wind pressure of specified intensity, generally between 30 and 50 lb. per sq. ft. acting over the vertical projected area from any direction.

However, if the structure is unusually large, precise information on the pressure distribution may be obtained by testing a model of the structure and its surroundings in a wind tunnel. Figure 98 shows the pressure distribution over the surface of a dirigible hangar as measured by Arnstein. The relatively

large percentage of the area which is under negative pressure is particularly noteworthy. Measurements by Dryden on a model of the Empire State Building showed that "the pressure varies from point to point, and that reduced pressure is found over the larger part of the model." It was also found that "a suitable value of the pressure for use in the design of tall buildings is $0.0038V^2$ (in pounds per square foot) where V is the wind speed in miles per hour against which provision is to be made." This value of dynamic pressure corresponds to an over-all force coefficient of 1.5.

OBJECTS AT A FREE SURFACE

91. Resistance of Ships.—The subject of ship resistance has been treated extensively both theoretically and experimentally and only a brief review of the basic relationships will be included here. For a more detailed treatment the student should consult "The Speed and Power of Ships" by D. W. Taylor.

The resistance of ships is caused by the generation of surface waves and submerged eddies and by skin friction. As the ship moves, the wave system is being continually elongated and the energy of this wave system can be supplied only by the ship itself. The time rate of increase of energy in the wave system equals the velocity times the wave-making resistance. Eddies form behind appendages and surface irregularities, and skin friction exerts a dragging force over the whole wetted surface. The subject of ship resistance is of special importance because it is the outstanding example of the successful use of models for the solution of flow problems in which gravity and viscosity are both important.

The resistance of a ship is known from experience to depend upon the following variables:

Total resistance......
Speed..............
Size...............
Density of fluid......
Viscosity............
Weight per unit mass
Geometrical ratios... [1]

Applying the Π-theorem to obtain the dimensionless groups and substituting ν for

$$R = \rho L^2 V^2 f\left(\frac{VL}{\nu}, \frac{Lg}{V^2},\right.$$

The unknown function f is a dimensionless coefficient of resistance which will have the same value whenever the dimensionless groups have the same values in two geometrically similar systems. If ν and g are the same in both systems, the requirement for similarity is that $V \propto 1/L$ and $V \propto \sqrt{L}$ which can be satisfied only if the two systems are of the same size.

The practical solution of this dilemma lies in the fact that a ship hull is a long, gradually curved surface on which the purely frictional drag is very nearly the same as on a flat surface moved parallel to itself. The eddy resistance is approximately independent of the viscosity and Reynolds number, while the wave-making resistance is nearly the same as in a frictionless liquid. These conditions lead to the assumption that the resistance may be broken up into two independent parts as

$$R =$$

The method of applying this equation to the prediction of ship resistance from model tests is as follows:

1. The total resistance of the model is measured experimentally as a function of velocity.

2. The frictional resistance is computed as a function of velocity on the basis of independent tests on flat plates of approximately the same length and wetted area as the model.

3. The wave-making resistance of the model, R_{wm}, is obtained at each velocity by subtracting the frictional resistance from the total resistance.

4. The wave-making resistance of the ship is then predicted for corresponding speeds as follows:

$$V = \qquad \frac{}{L} = \text{corresponding speed}$$

5. The total resistance of the ship is then obtained by adding the frictional resistance of a flat plate of the same length and wetted area as the hull and at the corresponding velocity.

This method yields results which are in such good agreement with full-scale measurements that the testing of models is a standard procedure in the design of ships. A refinement of the method which is applied to large models consists in reproducing the propellers as well as the hull and driving them at the corresponding speed.

Figure 99 shows a typical curve of ship resistance as measured by means of a model without propellers. The wave-making resistance is the difference between the total resistance and the skin friction. The curve of total resistance shows a decided "hump" which corresponds to a change in the wave pattern.

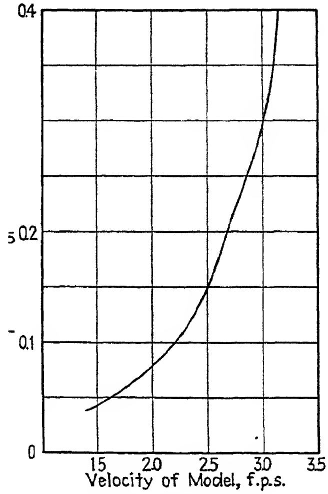


FIG. 99.—Typical curve of ship resistance.

The fact that surface tension sets a lower limit for the velocity of surface waves should be considered in testing small ship models. Since this limiting wave velocity is 0.76 ft. per sec. for water in contact with air, model velocities should be considerably greater if the wave-making resistance is to correspond to that of the full-scale ship. Since $V \propto \sqrt{L}$, there is evidently a lower limit to the scale reduction.

92. Stationary Objects in Streams.—Bridge piers, piles, and other objects which pierce the free surface of a stream offer a resistance to flow which is very difficult to predict. In addition to the form resistance which would occur if the same object were deeply submerged, there is a wave-making resistance similar to that of ships. Usually, the exact force on the pier is of less interest than its effect on the backwater curve upstream and the scour around it. For this reason, the forces exerted have not been investigated. Figure 100 shows the water-surface elevations around a circular cylinder towed through still water

as measured by Rehbock. Figure 101 shows the scour around

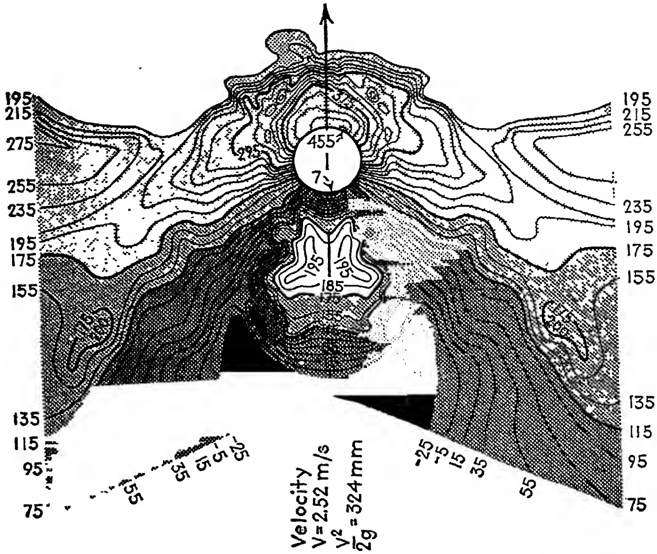


FIG. 100.—Surface contours around a circular cylinder towed through still water. ("*Hydraulic Laboratory Practice*," *A.S.M.E.*, 1929, *p.* 275.)

a circular pier as observed by Kopp. As the resistance coefficient will depend largely on the shape and spacing of the piers, and probably on the ratio of velocity head to depth, estimates of the resistance are difficult and whenever such resistance is of particular importance, the most satisfactory procedure is to study the flow conditions by means of a model. Yarnell found that data on the drop in surface elevation produced by a line of bridge piers could be represented by the formula

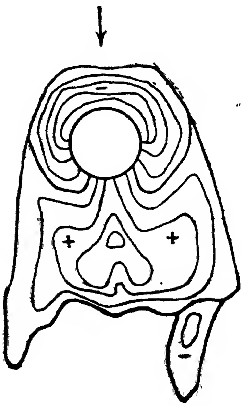


FIG. 101.—Scour around a circular pier.

$$\Delta H = 2K(K + 10\omega - 0.6)(\alpha +$$

Here K is a coefficient depending on shape, α is the percentage reduction in width, ω equals $V_3^2/2gD_3$, and V_3 and D_3 are the average velocity and

depth downstream. The formula is valid so long as the critical depth is not reached between the piers, (see Chap. IX, page 278). Values of K are as follows:

	K
Nose and tail semicircular.......	0.95
Nose and tail 90-deg. triangular.	1.05
Nose and tail square............	1 25

The same author obtained similar results for the resistance of pile trestles.

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Problems

1. A smooth plate 4 ft. wide and 8 ft. long is to be submerged in a stream of water having a velocity of 5 ft. per second. Compute the total frictional force if the plate is placed parallel to the direction of flow. Compute the force if the plate is placed transverse to the flow.

2. A wire $\frac{1}{4}$ in. in diameter is exposed to a stream of standard air at a velocity of 150 m.p.h. Compute the drag per foot length of wire.

3. A sphere 5 in. in diameter and weighing 10 lb. falls through the atmosphere from a considerable height. What is the terminal velocity of fall?

4. A spherical grain of quartz sand is 0.01 in. in diameter. Compute the velocity of fall in water at 75 deg. Fahr. given by Stokes' equation and compare with the velocity given by the equation at the bottom of page 189.

5. A rectangular airfoil has a span of 100 ft. and a chord of 10 ft. The airfoil section corresponds to NACA 6312 (see TR 460, National Advisory Committee for Aeronautics). Compute the lift and drag at an angle of attack of 3 deg. and an air speed of 100 m.p.h. (Proper corrections must be made for the aspect ratio.)

6. A model ship built to a scale of 1:100 is towed at a velocity of 3.20 ft. per sec. The total force is 0.368 lb. The length of the model is 3.95 ft. and the wetted area of the hull 3.538 sq. ft. Estimate the wave-making resistance of the model and the total resistance of the prototype at the corresponding speed.

CHAPTER VII

STEADY FLOW IN HYDRAULIC PIPELINES

In Chap. IV, it was shown that data on fluid friction could be generalized by the use of the Reynolds number so as to be applicable to all homogeneous fluids. This criterion of dynamical similarity for submerged flow applies to losses in elbows, valves, diverging and converging sections, meters, and all the other elements which may be included in pipelines and, consequently, data obtained with one fluid should be applicable to others. Unfortunately this method of generalization was unknown to, or disregarded by, the great majority of experimenters working on hydraulic friction losses with the result that much of the data is presented as being applicable to water alone.

As pipelines for transporting water are more numerous than any other type, the present chapter will consider them specifically and the methods followed will be those normally used in hydraulics. However, most of the data for losses in fittings were obtained in the region of fully developed turbulence where the coefficient of resistance is approximately constant and the results are applicable to other liquids of small viscosity, provided that the velocities and diameters are comparable with those in the experimental range with water.

The methods used here also apply to the flow of compressible fluids provided that the over-all pressure drop is small as compared with the absolute pressure and that the thermal effects are negligible. The error involved in applying the hydraulic equations to compressible fluids increases with the percentage change in density. In spite of what may be said for the accuracy of most hydraulic formulas, the general run of computations of hydraulic friction losses involves a probable error of at least ± 5 per cent and this fact should be considered in placing restrictions on the use of hydraulic equations for compressible flows. The effect of the compressibility is not random, however, and a correction may be made after the approximate pressure

distribution has been obtained from the hydraulic formulas. If the compressibility effects are appreciable, the methods used must be modified in a basic manner in that the sequence of losses must be considered. For example, if a certain weight of air is to be passed through a duct containing an orifice which causes a large *percentage drop in the absolute pressure,* the position of the orifice in the duct will obviously have an effect on the flow. Thermal effects may also be important even with negligible pressure drops and here again the sequence of changes must be considered for the flow conditions will depend upon the point at which the heat is added or removed. Even when the over-all

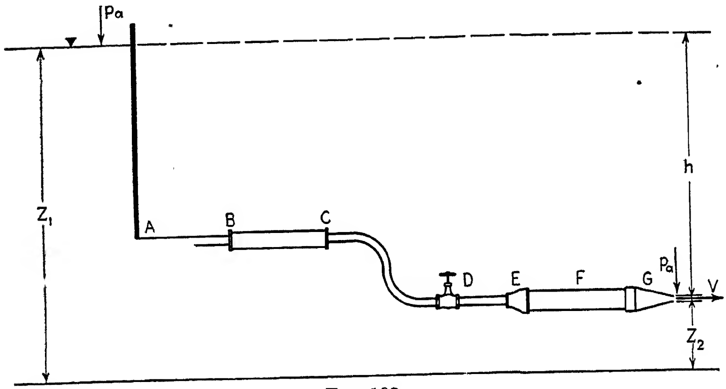

FIG. 102.

effects of compressibility or heat transfer are appreciable, many problems can be solved using the hydraulic equations by breaking the flow system up into sections in which the flow may be considered as hydraulic.

93. Losses in Pipelines.—Considering the pipeline shown in Fig. 102, the loss of mechanical energy by friction and impact is obtained from the relationship

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} + h_L. \tag{7.1}$$

where h_L denotes the mechanical energy lost. But

$$= p_2 = p_a = 0, \qquad V_1 = 0, \qquad z_1 - z_2 =$$

so

$$h_L = h - \frac{V_2^2}{2g} \tag{7.2}$$

This equation states that each unit weight of fluid which passes through the line has an initial potential energy equal to h, a final kinetic energy equal to $V^2/2g$, and that the difference in the two represents a decrease in mechanical energy. This decrease which is usually referred to as the *loss of head* or *head loss* for the piping system of Fig. 102 is made up of the following individual

1. Entrance at A.
2. Friction between A and B.
3. Expansion at B.
4. Friction between B and C.
5. Contraction at C.
6. Wall friction and bend losses between C and E.
7. Loss in the fitting at D.
8. Expansion at E.
9. Friction between F and G.
10. Friction in nozzle at G.

Dynamical similarity indicates, and experiment confirms the conclusion, that all these losses can be expressed by equations of the type, $h_L = K\frac{V^2}{2g}$, where the dimensionless coefficient K is a function of the Reynolds number. This method of representing the losses is normally applied only to turbulent flow in which case the coefficient K is approximately constant. In laminar flow, K is very nearly proportional to V^{-1}. The solution of flow problems in such systems then resolves itself into a determination of the proper values of K. The general equation for the loss of head becomes

$$h_L = K_1\frac{V_1^2}{2g} + K_2\frac{V_2^2}{2g} + \cdots + K_n\frac{V_n^2}{2g} = h - \frac{V^2}{2g} \qquad (7.3)$$

where V is the final velocity of discharge from the system, which may or may not be the same as V_n. If A_1 is the discharge area corresponding to the velocity V_1, and so forth,

$$AV = A_1V_1 = A_2V_2 = \cdots = A_nV_n$$

Substituting these equations in Eq. (7.3) gives

$$h = \frac{V^2}{2g} + K_1\left(\frac{A}{A_1}\right)^2\frac{V^2}{2g} + K_2\left(\frac{A}{A_2}\right)^2\frac{V^2}{2g} + \cdots + K_n\left(\frac{A}{A_n}\right)^2\frac{V^2}{2g} \qquad (7.4)$$

Solving for V,

$$V = \sqrt{\frac{\qquad}{1 + K_1\left(\frac{A}{A_1}\right)}} \tag{7.5}$$

and

$$= AV = A\sqrt{\frac{2gh}{1 + \qquad K_n\left(\frac{A}{A}\right)}}$$

For the general case of flow between any two points under different pressures, the quantity h may be written as [Eqs. (7.1) and (7.2)]

$$\left.\frac{p_2}{w} + z_2\right) = h_L + \frac{V^2}{2g}$$

Here, h is the equivalent head in terms of a column of fluid of unit weight w, which is that of the fluid flowing. The terminal velocity V is simply the velocity existing at the point at which the pressure is p_2 and the elevation z_2, and in the solution of this problem it is immaterial whether the corresponding velocity head is lost or not.

One difficulty in solving actual problems, even when sufficient experimental data are available, usually arises from the fact that the value of K cannot be selected until the various values of V are known and these in turn depend upon K as Eq. (7.5) shows. The solution is usually obtained by "trial and error," selecting some consistent set of K's as a start, or by assuming values of Q or V and computing h from Eq. (7.4).

94. Energy and Hydraulic Grade Lines.—Since Bernoulli's equation gives the energy per unit weight which is equal to a distance or length, the equation can be represented graphically by plotting these distances above an assumed datum plane. For the ideal flow without friction or impact losses, the *energy grade line* is horizontal but for any real flow it must drop continuously in the direction of motion. The elevation of the energy grade line at any point is $\left(z + \frac{p}{w} + \alpha\frac{V_m^2}{2g}\right)$. If atmospheric pressure is taken as the zero of pressure, the energy grade

line lies above the point considered by the amount $\left(\frac{p}{w} + \alpha\frac{V_m}{2g}\right.$. Here $V_m = Q/A$ and α is a coefficient representing the variation in velocity.

The locus of the distances p/w, plotted above the points considered, is known as the *hydraulic grade line.* With atmospheric pressure as datum the hydraulic grade line lies above the point if the pressure head is positive and below it if negative. When the hydraulic grade line is below the point by a distance equal to the barometric pressure head minus the vapor pressure head (both expressed in feet of the liquid flowing), the liquid column is under a negative pressure and cavities will form. Plotting the hydraulic grade line is particularly convenient in checking a pipeline for such conditions.

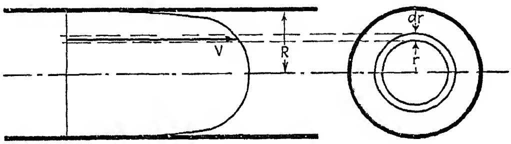

FIG. 103.

The energy and hydraulic grade lines are shown in Fig. 107.

In plotting the energy and hydraulic grade lines for vertical or inclined pipes, it is convenient to develop the line by plotting the elevation against the length along the center line. For horizontal, vertical, or inclined pipes, the hydraulic grade line has the same elevation for all points in a section perpendicular to the pipe axis since $\left(\frac{p}{w} + z\right)$ is the same for points so located.

95. Velocity Head.—Since the velocity of flow varies across a pipe, the true elevation of the energy grade line is different for each point. For the purposes of pipeline computations, however, it is the average velocity head which is important and for the great majority of problems this can be taken as $V_m^2/2g$. However, if the change in velocity head is of the same order as the change in $\left(\frac{p}{w} + \right.$ a more precise estimate of the average velocity may be necessary.

Assuming that the curve representing the velocity distribution at a cross section (Fig. 103) is symmetrical about the longitudinal

axis of the pipe, the mass of the fluid carried across an annular area of width dr in unit time is

$$\frac{W}{g} = 2\pi r \frac{w}{g} V\,dr$$

The kinetic energy passing the section in unit time is

$$\int_0^R 2\pi r \frac{w}{g} \frac{V^3}{2}\,dr$$

The mean velocity head is the average kinetic energy per unit weight and can be expressed as

$$\left(\frac{V^2}{2g}\right)_m = h_v = \frac{\int_0^R 2\pi \frac{w}{g} \frac{V^3}{2} r\,dr}{\int_0^R 2\pi w V r\,dr} = \frac{\pi}{g} \frac{\int_0^R V^3 r\,dr}{Q} = \frac{\pi}{g} \frac{\int_0 V^3 r\,dr}{}$$

Since the mean velocity is readily determined, it is convenient to express h_v as

where $\alpha > 1$. The equation defining α is then

$$\alpha - \frac{}{A V_m^3}$$

The summation indicated can best be performed graphically by plotting $V^3 r$ against r and planimetering the area under the curve. The value of α is usually about 1.1 when the distribution of velocity is determined by friction alone, and may be neglected in most cases. However, the value of α may be much greater when the flow is disturbed, as by baffles or immediately after bends or valves, and should be considered if the change in

is small. The value of α is related to the ratio of the mean to the maximum velocity and since this ratio varies only from 0.5 to about 0.85 in most cases of flow in pipes, the variation of α is ordinarily not great. The value of α may also be obtained for asymmetrical velocity distributions by similar graphical methods.

The effect of converging boundaries is to tend to produce a uniform velocity distribution and hence to make α approach unity. If the velocity distribution in the approach section is determined by friction, α is greater than unity and the theoretical flow equation between two such points is approximately

$$\frac{V_{2m}^2 - \alpha_1 V_{1m}^2}{2g} = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)$$

The change in velocity head is equal to the change in $\left(\frac{p}{w} + z\right)$ and the effect of α is not necessarily negligible. In devices such as the venturi meter and the flow nozzle, the effect of α is included in the coefficient of discharge.

In the remainder of this chapter, the mean velocity will be indicated by V without the subscript and α will be assumed as unity.

96. Friction Losses in Straight Circular Pipe.—The Reynolds number method of representing friction coefficients was considered in Chap. IV and it was found that over limited ranges of Re, the friction coefficient should be expressed as

$$f = m\mathrm{Re}^n$$

Since most of the experiments on water covered only a narrow range of Re, the results when plotted on a logarithmic diagram gave straight lines for any one set of experiments but the values of m and n varied not only with the relative roughness but also with the range of Re covered by the experiments. For reference a few of these empirical formulas are mentioned below.

1. *Chezy-Darcy Equation.*—In 1775, Chezy proposed the formula

$$V = C\sqrt{RS}$$

where V = the mean velocity of flow.
C = a coefficient.
R = the hydraulic radius.
S = the slope of the energy gradient.

In 1857 Darcy modified this equation by inserting $h_L/L = S$, and $D = 4R$. Solving for h_L

$$h_L = f\frac{L}{D}\frac{V^2}{2g} = K\frac{V^2}{2g}$$

where

$$f = \frac{8g}{C^2}, \qquad \text{or} \qquad C = 2\sqrt{\frac{2g}{f}}$$

This equation was also developed by Weisbach and is often named for him.

Darcy gave as the values of f:

$$f = 0.02 + \frac{0.02}{12D} \text{ (new, clean, cast-iron pipe)}$$

$$f = 0.04 + \frac{0.04}{12D} \text{ (old cast-iron pipe)}$$

Here, D is expressed in feet.

The values of f given by Darcy's formulas were found not to be generally applicable and Fanning presented his experimental values on straight smooth pipe in the form of a table, which showed the variation of f with both V and D. This method of representation is equivalent to the Reynolds number diagram for a constant viscosity. Several of these tables of the variation of f for water are included in Appendix II.

2. *Williams and Hazen Equation.*—The formula proposed by Williams and Hazen is similar to the Chezy equation:

$$V = C_0 R^{0.63} S^{0.54} 0.001^{-0.04}$$

The value of C_0 is said to depend only upon the roughness of the pipe walls. Average values are:

	C_0
Extremely smooth straight pipe	140
Smooth pipes	130
Riveted steel pipe	110

This equation has been widely used in spite of its curious form and is reliable within the usual range of conditions provided that the pipe can be properly classified as to roughness. The corresponding value of C in the Chezy formula is

$$\begin{aligned} C &= C_0 R^{0.13} S^{0.04} 0.001^{-0.04} \\ &= 1.318 C_0 R^{0.13} S^{0.04} \end{aligned} \tag{7.6}$$

By substituting for S and R, Eq. (7.6) may be reduced to

$$f = m'V^{-0.15}D^{-0.17},$$

which is approximately $f = m\mathrm{Re}^n$.

3. *Other Exponential Formulas.*—Unwin took as the basic equation

$$h_L = K\frac{L}{D^t}\frac{V^p}{2g} \tag{7.7}$$

and obtained the values of K, p, and t given in the following table:

Kind of pipe	K	t		$3 - t$
Wrought iron........	0.0226	1.21		1.79
Asphalted iron.......	0.0254	1.127		1.87
Riveted wrought iron	0.0260	1.390		1.61
New cast iron.......	0.0215	1.168		1.83
Cleaned cast iron....	0.0243	1.168	2.00	1.83
Incrusted cast iron...	0.0440	1.160	2.00	1.84

Rewriting Eq. (7.7) as

$$h_L = \frac{LV^2}{D2g}\,K\frac{V^{p-2}}{D^{t-1}}$$

the friction factor f is then $f = KV^{p-2}/D^{t-1}$. For dimensional homogeneity, $f \propto (VD/\nu)^n$ and therefore

$$n = p - 2 = 1 - t$$
$$p = 3 - t$$

This relationship for any pipe roughness is in a way a measure of the degree of correspondence between the Reynolds number equation and the formula. However, complete agreement is not to be expected. The exponential formula for f is correct only over limited ranges and the effect of roughness has not been included in the proper manner since K is an absolute coefficient for each character of pipe. The last two items in the table show $p = 2$, corresponding to fully developed turbulence. If the effect of roughness in these two cases is expressed as

the exponent $x = 0.16$. Additional exponential formulas have been developed by Scobey, Barnes, Lea, Manning, and others. These formulas are reliable for the particular types of surface tested but extrapolation to fluids other than water or to values of velocity and diameter outside the range of the original tests is dangerous for the reasons mentioned in the preceding paragraphs.

For the purpose of solving problems given in this text the friction factors will be assumed to be precisely those given in the tables or graphs. In any field problem, however, the difficulty lies not in the solution of the equations but in the proper choice of coefficients and judgment in this matter is gained only by experience.

This chapter deals specifically with flow in pipelines which are normally of circular section. For other shapes, the friction computations may be based on data for circular pipes by use of the hydraulic radius, which is a length equal to the area of the cross section divided by the perimeter. The hydraulic radius of a circular pipe is $D/4$. The corresponding Reynolds number is $4RV/\nu$. This method is not exact but is sufficiently precise for usual problems.

97. Losses in Fittings. 1. *Entrance Losses.*—When water enters a pipe from a reservoir, there is a loss at the entrance due to the contraction and subsequent enlargement of the stream. For an inwardly projecting pipe such as the Borda tube discussed on page 67 the theoretical loss is, by Eq. (3.22),

$$h_L = \left(\frac{1}{0.5} - 1\right)^2 \frac{V^2}{2g} = \frac{V^2}{2g}$$

and for a sharp-cornered entrance as shown in Fig. 102, where the coefficient of contraction is about 0.60 (Chap. V), the loss should be

$$h_L = \left(\frac{1}{0.6} - 1\right)^2 \frac{V^2}{2g} = 0.444\frac{V^2}{2g}$$

These expressions do not take into account friction losses, the energy represented by the turbulent secondary flow, or the distribution of velocity. Generally the entrance loss is expressed as

$$h_L = K_e\frac{V^2}{2g}$$

COEFFICIENTS OF ENTRANCE LOSS[1]

Type of entrance	K_e
Inward projecting.	0.78
Sharp-corner flush.	0.50
Slightly rounded...	0.23
Bellmouth.........	0.04

[1] H. W. King, "Handbook of Hydraulics."

2. *Sudden Contractions.*—The loss of head at an abrupt, sharp-edged contraction in a pipeline may be expressed as

$$h_L = K_c \frac{V^2}{2g}$$

where K_c is an experimentally determined coefficient and V is the velocity in the smaller pipe (after the contraction).

Values of K_c based on Weisbach's experiments are given below:

VALUES OF K_c FOR SUDDEN CONTRACTIONS IN PIPELINES

$m = (A_2/A_1)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
K_c	0.362	0.338	0.308	0.267	0.221	0.164	0.105	0.053	0.015

The phenomenon of abrupt contraction is similar to flow through a sharp-edged orifice or Borda tube. The loss coefficient, K_c, is a function of the geometry of the contraction. The values quoted apply to the region of fully developed turbulence and are sufficiently accurate for most pipeline problems in which other losses predominate.

3. *Abrupt Expansions.*—The head loss in an abrupt expansion has already been treated in Chap. III, page 61. Referring the head loss to the higher velocity,

$$h_L = (1 - m)^2 \frac{V^2}{2g} = K_a \frac{V^2}{2g}$$

where $m = A_1/A_2$. Experiments show a slight deviation from this formula. However, for a velocity of 4 ft. per sec. in the smaller pipe and $m = 0.25$, King gives the coefficient K_a as 0.56 which is identical with that given by the theoretical formula. The coefficient decreases with increasing velocity, and this

variation must be considered in problems in which the expansion loss is a large percentage of the total loss.

4. *Gradual Expansions.*—If the expansion from one diameter to a larger one is guided by solid boundaries, the loss of head due to shock is less than for an abrupt expansion provided that the expansion angle is not too great. However, as the angle included between the sides decreases, the length of the expanding section increases for a certain area ratio and therefore the friction loss increases with decreasing angles. There must then be a certain angle between the side walls at which the loss is a minimum. This angle was found experimentally by Gibson to be 5.5 deg.

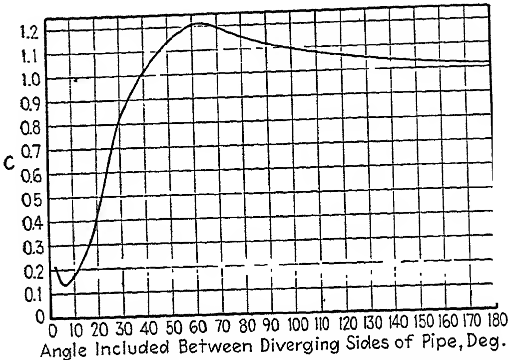

FIG. 104.—Coefficient of head loss in circular expanding pipes.

for circular sections for all area ratios and was not dependent on the size of pipe. The angle of minimum loss was 6 deg. for rectangular sections. Expressing the loss in diverging sections as

Gibson's experiments on circular, diverging sections show C to be 0.135 at 5.5 deg., and that between 7.5 and 35 deg., $C = 0.011\theta^{1.22}$, where θ is in degrees. The loss is a maximum at $\theta = 60$ deg. at which angle the value of C is 1.22 for $A_1/A_2 = 0.25$. At greater angles, the loss coefficient decreases and is very nearly unity at 180 deg.

5. *Nozzles.*—The loss of energy in nozzles may be expressed by the formula $h_L = K_N\frac{V^2}{2g}$, where V is the velocity of efflux. Values

of K_N as determined by Freeman for smooth nozzles are given below:

VALUES OF K_N FOR FIRE NOZZLES

Diameter of nozzle, in.	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$
K_N	0.037	0.038	0.051	0.058	0.061	0.087

The values stated show a scale effect and therefore an influence of Reynolds number if the nozzles were geometrically similar. For needle nozzles, K_N may be much greater, but no generally applicable values can be given. Energy losses in nozzles were discussed in Chap. V.

6. *Bends.*—The loss of head in pipe bends is a subject which has not been adequately investigated in spite of its importance

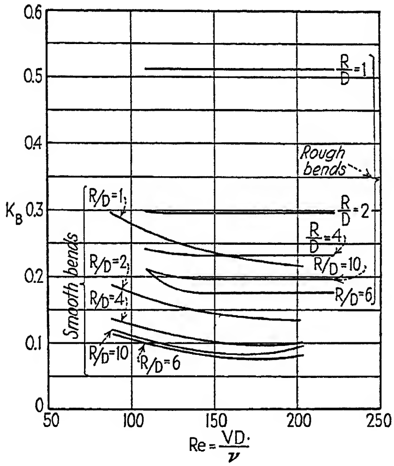

FIG. 105.—Coefficient of bend resistance.

in engineering computations. From a consideration of the conditions for dynamical similarity of flow, the experimental data for geometrically similar bends should be correlated on the basis of the Reynolds number and the roughness of the pipe walls, with the latter playing a minor part because of the essentially turbulent nature of the phenomenon. The additional variables needed to represent the data on all bends would appear to be

the angle of the bend and the ratio of the radius of curvature the diameter of the pipe.

The flow conditions at a bend are relatively complex, the high velocity filaments in the center of the approach pipe move to the outside of the bend, causing a secondary spiral flow. Below the bend, the velocity distribution is not symmetrical and a considerable length is necessary for a symmetrical distribution to be attained. Centrifugal forces and the displacement of the thread of maximum velocity make difficult an accurate

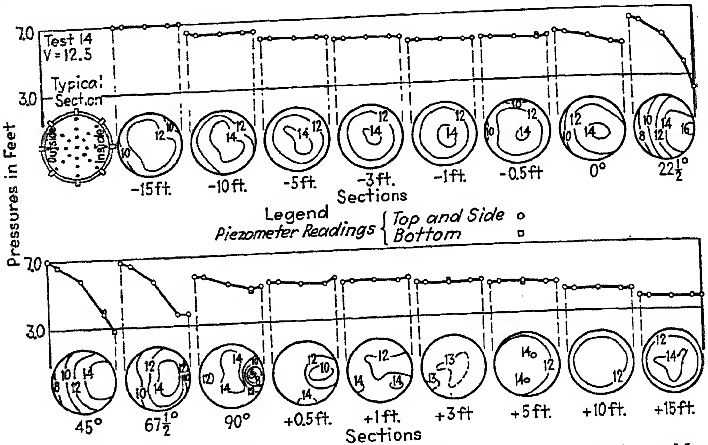

FIG. 106.—Velocity distribution at a 90-deg. bend in a 6-in. pipe. Mean velocity, 12.5 ft. per sec. (*D. L. Yarnell and F. A. Nagler*, "*Flow of Water around Bends in Pipes*," *Trans. A.S.C.E., vol.* 100, *p.* 1018, 1935.)

determination of the loss of head. Figure 106 shows the velocity distribution at a bend as measured by Yarnell and Nagler. Actual pipeline conditions are even more complicated since, in general, the fittings are of such design as to cause a sudden expansion entering the bend and a contraction leaving it.

There are two methods commonly used for the representation of data on the loss of head in bends. The first expresses the loss by means of the formula

$$h_L = \quad V^2$$

where V is the average velocity in the pipe and h_L is the loss in s of that for the same length of straight pipe. The second

method expresses the loss in terms of the length of straight pipe of the same diameter and roughness, which would produce the same loss at the same velocity. The loss in bends is therefore represented as a length to be added to the actual length of the pipeline for the purpose of computing hydraulic losses. The computations are then made as for a straight pipe using this effective length in place of the actual length. Values of K_B and the equivalent friction length of elbows and other screw fittings are to be found in Appendix II (Tables 16 and 17).

7. *Valves.*—Since valves are used for controlling flow by changing either the friction loss or the free discharge area, or both, and can be adjusted to give the desired flow, it is only in the wide-open position that the loss caused by valves is of much impor-

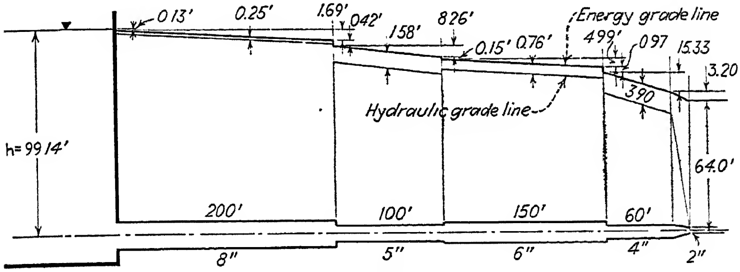

FIG. 107.

tance. Friction losses for wide-open globe and gate valves are given in Appendix II (Table 17).

PIPELINE PROBLEMS

98. Determination of Friction Losses for Known Rate of Flow and Dimensions of Pipeline.—A typical arrangement is shown in Fig. 107. The pipeline discharges freely into the atmosphere through a 2-in. nozzle having a coefficient $K_N = 0.05$, and is composed of straight sections of wrought-iron pipe. The rate of discharge is 1.40 cu. ft. per sec. The total head is given by the equation

$$h = K_e\frac{V_8^2}{2g} + \frac{f_8L_8}{D_8}\frac{V_8^2}{2g} + K_{c8\text{-}5}\frac{V_5^2}{2g} + \frac{f_5L_5}{D_5}\frac{V_5^2}{2g} + \frac{(V_5 - V_6)^2}{2g}$$
$$+ \frac{f_6L_6}{D_6}\frac{V_6^2}{2g} + K_{c6\text{-}4}\frac{V_4^2}{2g} + \frac{f_4L_4}{D_4}\frac{V_4^2}{2g} + K_N\frac{V_2^2}{2g} + \frac{V_2^2}{2g}$$

The subscripts indicate the pipe diameter. Using a table of standard pipe sizes (Appendix II, Table 12), the following values are obtained:

Nominal pipe size, in.	Actual pipe size, in.	Area, sq. ft.	$V = 1.40/A$, ft./sec.	$V^2/2g$
8	7.981	0.3474	4.03	0.252
5	5.047	0.1388	10.08	1.580
6	6.065	0.2006	6.98	0.758
4	4.026	0.0884	15.84	3.90
Nozzle.......	2.00	0.0218	64.2	64.0

From Appendix II, Table 14, the corresponding values of the friction factors are by interpolation:

$$f_8 = 0.0223, \qquad f_5 = 0.0220$$
$$f_6 = 0.0222, \qquad f_4 = 0.0220$$

The number of significant figures is greater than is warranted by the accuracy of pipe data but they are listed as given in the table, for identification. The coefficients of contraction are obtained from the data on page 211 as:

$$\text{Entrance:} \qquad K_e = 0.5$$
$$m_{8\text{-}5} = 0.400, \qquad \zeta_{c8\text{-}5} = 0.267$$
$$m_{6\text{-}4} = 0.441, \qquad \zeta_{c6\text{-}4} = 0.248$$

Substituting the numerical values, the equation becomes

$$h = 0.5 \times 0.252 + 0.0223 \times \frac{200 \times 12}{7.98} \times 0.252 + 0.267 \times 1.580$$
$$0.0220 \times \frac{}{5.05} \times 1.580 + \frac{}{64.4}$$
$$+ 0.0222 \times \frac{150 \times 12}{6.06} \times 0.758 + 0.248 \times 3.90$$
$$+ 0.0220 \times \frac{60 \times 12}{4.03} \times 3.90 + 0.05 \times 64.0 + 64.0$$
$$= 99.14 \text{ ft.}$$

Figure 107 shows the energy and hydraulic grade lines for this problem. If the pipeline had not been horizontal, the only effect would have been a change in the pressure distribution, the

two grade lines remaining unchanged if plotted against the developed length of the line.

99. Determination of the Discharge for a Known Head and Pipe Size.—This problem may be solved using the methods of the preceding section by assuming a rate of discharge and computing the corresponding head. This first assumption will not give a head equal to that specified and a second assumption is necessary which can be made by utilizing the fact that the losses are very nearly proportional to the square of the discharge. If h' is the computed head corresponding to the assumed discharge Q', then the approximate discharge Q'', for the specified head h, is roughly $Q'' = Q'\sqrt{h/h'}$ Computation of the head for this

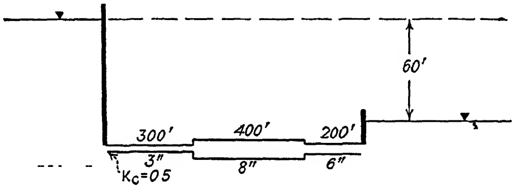

FIG. 108.

discharge gives a head h'' which is very nearly equal to h. If the two points (Q', h') and (Q'', h'') are plotted on logarithmic paper and a straight line drawn between them, the correct discharge, Q can be obtained by interpolation.

The trial-and-error method is frequently used in solving this type of problem and an example will be worked out on this basis. The piping system is assumed to be made up of standard wrought-iron pipe arranged as shown in Fig. 108. The equation for the head loss is

$$h = 60' = K_e\frac{V_3^2}{2g} + \frac{f_3L_3}{D_3}\frac{V_3^2}{2g} + \left(1 - \frac{A_3}{A_8}\right)^2\frac{V_3^2}{2g} + \frac{f_8L_8}{D_8}\frac{V_8^2}{2g} + K_{c8\text{-}6}\frac{V_6^2}{2g} + \frac{f_6L_6}{D_6}\frac{V_6^2}{2g} + \frac{V_6^2}{2g}$$

Substituting for V in terms of Q and A, inserting numerical values, and simplifying,

$$Q = \frac{62.2}{\sqrt{500 + 446{,}000f_3 + 5000f_8 + 9800f_6}}$$

Values of f must be assigned before the problem can be solved and these values can be chosen accurately only if the velocity is known. For the

purpose of choosing the friction factors, assume $V_3 = 16$ ft. per sec. Then the corresponding values obtained from Appendix II, Table 14, are

$$V_8' = 2.25 \text{ ft. per sec.,} \qquad f_8' = 0.0242$$
$$V_6' = 4.0 \text{ ft. per sec.,} \qquad f_6' = 0.0240$$
$$f_3' = 0.0238$$

The corresponding discharge is 0.58 cu. ft. per sec. and the velocity in the 3-in. pipe is 11.3 ft. per sec. Selecting new friction coefficients on the basis of this first approximation,

$$V_3'' = 11.3 \text{ ft. per sec.,} \qquad f_3'' = 0.0247$$
$$V_8'' = 1.59 \text{ ft. per sec.,} \qquad f_8'' = 0.0254$$
$$V_6'' = 2.82 \text{ ft. per sec.,} \qquad f_6'' = 0.0252$$
$$Q'' = 0.569 \text{ cu. ft. per sec.,} \qquad V_3'' = 11.1 \text{ ft. per sec.}$$

Using the result of the second approximation,

$$V_3''' = 11.1 \text{ ft. per sec.,} \qquad f_3''' = 0.0248$$
$$V_8''' = 1.56 \text{ ft. per sec.,} \qquad f_8''' = 0.0255$$
$$V_6''' = 2.78 \text{ ft. per sec.,} \qquad f_6''' = 0.0253$$
$$Q''' = 0.568 = Q$$

The third approximation yields a result differing from the second by a negligible amount and the problem is solved.

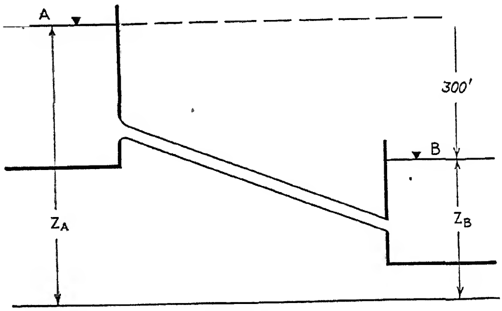

FIG. 109.

100. Determination of the Pipe Diameter for a Known Head and Discharge.—As an example, the pipe diameter in Fig. 109 will be chosen so as to deliver 1800 gal. per min. through 4000 ft. of new cast-iron pipe under a head of 300 ft.

Writing Bernoulli's equation between the two reservoirs,

$$\frac{p_A}{w} + z_A + \frac{V_A^2}{2g} = \frac{p_B}{w} + z_B + \frac{V_B^2}{2g} + h_L$$

But $p_A = p_B = 0$, and V_A and V_B may be neglected.

Hence

$$h_L = z_A - z_B = 300 = K_e\frac{V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g} + \frac{V^2}{2g}$$

For a bellmouth entrance $K_e = 0.04$ (from page 211). Inserting the numerical values,

$$300 = \left(1.04 + f\frac{4000}{D}\right)\frac{V^2}{2g}, \qquad Q = \frac{1800}{449} = 4.01 \text{ cu. ft. per sec.}$$

$$V = \frac{Q}{A} \qquad 300 = \left(1.04 + f\frac{4000}{D}\right)\frac{Q^2}{2g\left(\frac{\pi D^2}{4}\right)^2}$$

The solution of this equation involving two unknowns, f and D, is best made by trial and error, assuming different pipe diameters until one is found which fits the conditions of the problem.

$$\sqrt{\frac{2g \times 300 \times \pi^2 \times D^5}{16(1.04D + 4000f)}} \qquad \sqrt{1.04D + 4000f}$$

Nominal pipe diameter, in.	Pipe diameter, ft.	Pipe area, sq. ft.	V, ft./sec.	f, Table 14, Appendix II	Q, cu. ft./sec.
7	0.583	0.267	15.02	0.0193	3.21
8	0 667	0.349	11.50	0.0193	4.49
9	0.750	0.442	9.07	0.0195	6.00

It appears from the tabulated quantities that an 8-in. pipe will be large enough to carry the desired quantity of water. The exact diameter may be obtained graphically if standard pipe sizes are not required.

101. Power Obtainable from a Pipeline.—The power obtainable from a pipeline is the product of the weight of water discharged per unit time and the energy per unit weight possessed by the water as it leaves the pipe. If the maximum possible quantity of water is allowed to flow through a given pipe system and discharge freely into air, a large part of the initial energy content will be used up in friction, and the available power will be small. On the other hand, if the velocity of the water in the pipe be greatly reduced by a nozzle at the end, the available energy per unit weight will be increased but the quantity may be so small that the available power is again small. It would seem that there is an optimum size of nozzle for a given pipeline which

will deliver the maximum power. The power delivered for any arrangement is

$$P = Qw\frac{V_j^2}{2g} \tag{7.8}$$

where V_j is the average velocity of the jet. From Eq. (7.4),

$$\frac{V_j^2}{2g} = h - \left[K_e\frac{V_p^2}{2g} + f\frac{L}{D}\frac{V_p^2}{2g} + K_N\frac{V_p^2}{2g}\left(\frac{A_p}{A_j}\right)^2\right] = h - K\frac{V_p^2}{2g}$$

where

$$K = K_e + f\frac{L}{D} + K_N\left(\frac{A_p}{A_j}\right)^2$$

Substituting in Eq. (7.8),

$$P = Qw\left(h - K\frac{V_p^2}{2g}\right) = wA_p\left(hV_p - K\frac{V_p^3}{2g}\right)$$

Differentiating with respect to V_p and assuming K to be constant, which is very nearly correct,

$$\frac{dP}{dV_p} = w$$

But $KV_p^2/2g = h_L$, the loss of head in the pipeline and nozzle, and hence the maximum power output occurs when the loss of head is one-third of the total available head. At this condition,

$$V_j = \sqrt{2g\tfrac{2}{3}h}, \qquad P = wA_j\frac{V_j^3}{2g}$$

The ratio of nozzle diameter to pipe diameter to give maximum power is found from these equations to be

$$\frac{A_p}{A_j} = \sqrt{\frac{2\left(K_e + f\frac{L}{D}\right)}{1 - 2K_N}}$$

It should be mentioned in connection with this solution that the maximum power output is not necessarily the desirable operating condition since the design of a pipeline is based upon the relationship between the capitalized value of the power lost and the cost of building a larger pipeline.

102. Parallel Pipes.—In some distributing mains the flow is divided between two or more pipes which later rejoin. On the assumption that the friction loss in the lines is *large in comparison with the loss at the branch points,* the problem can be solved on the basis of the friction loss in the individual lines and the equation of continuity.

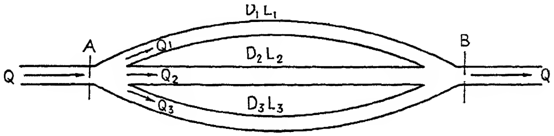

FIG. 110.

Considering the system shown in Fig. 110, the total head lost between points A and B is the same for all paths, and hence the set of equations for any number of parallel lines is

$$h_L = K_1\frac{V_1^2}{2g} = K_2\frac{V_2^2}{2g} = K_3\frac{V_3^2}{2g} = \cdots = K_n\frac{V_n^2}{2g}$$

Here, K is a coefficient representing all losses along the path. If losses in elbows and other fittings are negligible as compared with the pipe friction loss, these equations would be

$$h_L = \frac{f_1L_1}{D_1}\frac{V_1^2}{2g} = \frac{f_2L_2}{D_2}\frac{V_2^2}{2g} = \frac{f_3L_3}{D_3}\frac{V_3^2}{2g} = \cdots = \frac{f_nL_n}{D_n}\frac{V_n^2}{2g}$$

It should be noted that these, as well as all preceding friction equations, apply in viscous as well as turbulent flow provided that f is properly chosen. In the laminar region, $f = 64/\text{Re}$ for circular pipes and the head loss is proportional to the first power of the velocity. The continuity equation is

$$Q = A_1V_1 + A_2V_2 + A_3V_3 + \cdots + A_nV_n$$

Rearranging the equations,

$$= V_2\sqrt{\frac{K_2}{K_1}} = V_3\sqrt{\frac{K_3}{K_1}} = \cdots = V_n\sqrt{\frac{K_n}{K_1}}$$

$$V_1\left(A_1 + A_2\sqrt{\frac{K_1}{K_2}} + A_3\sqrt{\frac{K_1}{K_3}} + \cdots + A_n\sqrt{\frac{K_1}{K_n}}\right)$$

The difficulty about solving the equation is that the values of K are dependent on the values of f and are therefore not known

until the velocities are known. Solution by trial and error, assuming values of f as in other friction problems, quickly leads to the proper solution.

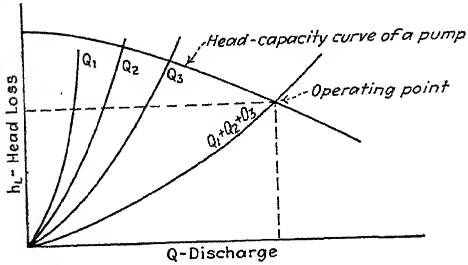

FIG. 111.

A more straightforward method consists in computing the characteristic curves for each of the parallel lines by assuming values of Q and then drawing a curve of ΣQ for constant values of h_L. This method is illustrated by Fig. 111, in which the

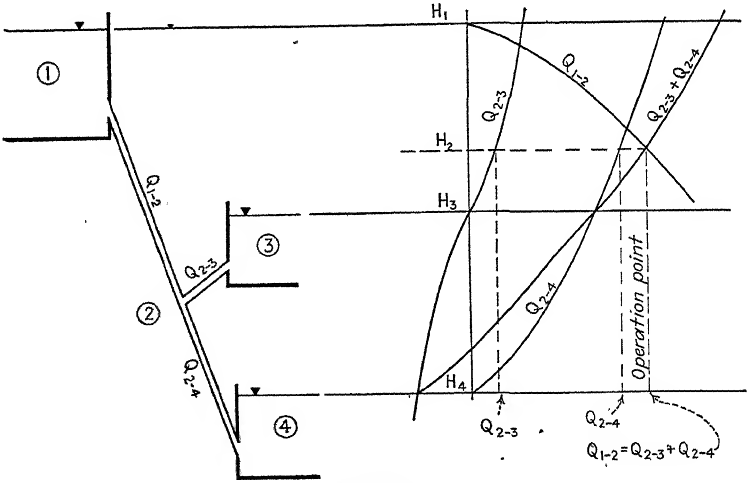

FIG. 112.

head-capacity curve for a pump or blower has been inserted. The total discharge of the pump is fixed by the intersection of the characteristic curve with the curve of ΣQ. The flow in each line is then found at constant h_L as shown.

103. Branching Pipes (Three-reservoir Problem).—The determination of the flow in pipelines joining three or more reservoirs may be solved by algebraic or graphical means, employing the usual friction equations as the basis. It will be assumed that the velocity heads in the lines and the losses of head at the point of division are small as compared with the friction losses.

Referring to the left-hand portion of Fig. 112, the direction of flow in the line 2-3 will depend upon the elevation of the hydraulic grade line at (2), which in turn depends upon the relative friction losses in the lines 1-2 and 2-4. The first step in the solution is then to assume the line 2-3 to be closed and to compute the corresponding elevation H_2. If H_2 is greater than H_3, flow will occur into the reservoir (3) even after the line 2-3 is opened, and vice versa. For flow into reservoir (3), the equations are

$$Q_{1\text{-}2} = Q_{2\text{-}3} + \qquad (7.9a)$$

$$H_1 \qquad _2 = K_{1\text{-}2} \frac{Q_{1\text{-}2}}{2gA_{1\text{-}2}} \qquad (7.9b)$$

$$_3 = K_{2\text{-}3} \frac{Q_{2\text{-}3}{}^2}{2gA_{2\text{-}3}} \qquad (7.9c)$$

$$H_2 - H_4 = K_{2\text{-}4} \qquad (7.9d)$$

For flow out of reservoir (3), Eqs. (7.9) are modified as follows:

$$Q_{1\text{-}2} + Q_{3\text{-}2} = Q_{2\text{-}4} \qquad (7.10a)$$

$$H_3 - H_2 = \frac{K_{2\text{-}3}Q_{3\text{-}2}{}^2}{\quad 2} \qquad (7.10b)$$

1. *Graphical Solution.*—Figure 112 illustrates the graphical method of solution. Values of $Q_{1\text{-}2}$, $Q_{2\text{-}3}$, and $Q_{2\text{-}4}$ are assumed and plotted on the diagram against the value of H_2 computed from Eqs. (7.9) and (7.10). The summation curve is drawn with proper regard to the fact that for $H_2 > H_3$, Eq. (7.9a) applies while for $H_2 < H_3$, Eq. (7.10a) gives the continuity relationship. The intersection shown in Fig. 112 corresponds to flow into the reservoir (3).

2. *Algebraic Solution.*—Referring to Fig. 113, it is necessary to determine the distribution of flow in the lines connecting the three reservoirs. The lines will be assumed to be made up of new wrought-iron pipe.

Closing the 5-in. line to determine the direction of flow in that line,

$$100 = 0.5\frac{V_6^2}{2g} + f_6\frac{500 \times 12}{6.06}\frac{V_6^2}{2g} + f_8\frac{1000 \times 12}{7.98}\frac{V_8^2}{2g} + \frac{V_8^2}{2g}$$

Substituting for V_6 and V_8 by $V = Q/A$,

$$100 = 0.5\frac{Q^2}{2g \times 0.201^2} + f_6\frac{500 \times 12 \times Q^2}{6.06 \times 2g \times 0.201^2} + f_8\frac{100 \times 12 \times Q^2}{7.98 \times 2g \times 0.347^2} + \frac{Q^2}{2g \times 0.347^2}$$

$$Q = \sqrt{\frac{2g \times 100}{12.4 + 24{,}700f_6 + 12{,}500f_8 + 8.3}}$$

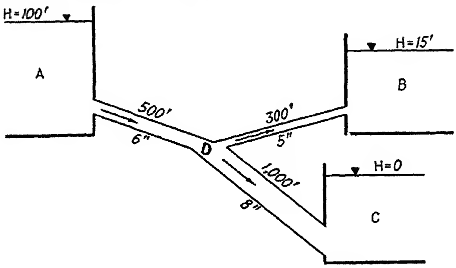

FIG. 113.

Estimating V_8 at 10 ft. per sec.,

$$V_6 = 10 \times \frac{8^2}{6^2} = 17.8 \text{ ft. per sec.}$$

$$f_8 = 0.0197, \qquad f_6 = 0.0195,$$

$$Q = \sqrt{\frac{6440}{20.7 + 482 + 246}} = 2.94 \text{ cu. ft. per sec.}$$

Checking velocities,

$$V_8 = \frac{2.94}{0.347} = 8.49 \text{ ft. per sec.}$$

and

$$V_6 = \frac{2.94}{0.201} = 14.6 \text{ ft. per sec.}$$

Recalculating,

$$f_8 = 0.0201, \qquad f_6 = 0.0201,$$

$$Q = \sqrt{\frac{6440}{20.7 + 496 + 251}} = 2.88 \text{ cu. ft. per sec.}$$

Rechecking velocities,

$$V_8 = \frac{2.88}{0.347} = 8.30 \text{ ft. per sec.}$$

and

$$V_6 = \frac{2.88}{0.201} = 14.3 \text{ ft. per sec.}$$

This is close enough for a first approximation to the hydraulic gradient at D, which is

$$\frac{p_0}{w} + z_0 = 100 - \left(0.5\frac{14.3^2}{2g} + 0.0201\frac{500 \times 12}{6.06}\frac{14.3^2}{2g} + \frac{14.3^2}{2g}\right)$$
$$= 100 - (1.6 + 63.2 + 3.2) = 32.0 \text{ ft.}$$

The velocities are sufficiently high that the losses at the branch point will be appreciable. The flow evidently will occur from D toward B and the elevation of the energy grade line just upstream from the division with the 5-in. line open is less than

$$E_D = 32.0 + 3.2 = 35.2 \text{ ft.}$$

To obtain a first approximation to the flow distribution, it will be assumed that no shock loss occurs at the point of division. On this basis, the energy equations for the three lines are

$$(AD)\ E_D = 100 - 0.5\frac{V_6^2}{2g} - f_6\frac{500 \times 12}{6.06}\frac{V_6^2}{2g}$$
$$(DB)\ E_D - f_5\frac{300 \times 12}{5.04}\frac{V_5^2}{2g} - \frac{V_5^2}{2g} = 15$$
$$(DC)\ E_D - f_8\frac{1000 \times 12}{7.98}\frac{V_8^2}{2g} - \frac{V_8^2}{2g} = 0$$

These equations simplify to

$$V_6 = \sqrt{\frac{2g(100 - E_D)}{0.5 + 990f_6}}$$
$$V_5 = \sqrt{\frac{2g(E_D - 15)}{714f_5 + 1}}$$
$$V_8 = \sqrt{\frac{2gE_D}{1500f_8 + 1}}$$

Since E_D must be less than 35.2 ft., assume $E_D = 30$ ft., and also assume values of $f_6 = 0.0200$, $f_8 = 0.0200$, $f_5 = 0.0250$.

$$V_6 = \sqrt{\frac{2g \times 70}{0.5 + 19.8}} = 14.9 \text{ ft. per sec.}$$
$$Q_6 = 14.9 \times 0.201 = 3.00 \text{ cu. ft. per sec.}$$
$$V_5 = \sqrt{\frac{2g \times 15}{17.8 + 1}} = 7.18 \text{ ft. per sec.}$$
$$Q_5 = 7.18 \times 0.139 = 0.996 \text{ cu. ft. per sec.}$$
$$V_8 = \sqrt{\frac{2g \times 30}{30.0 + 1}} = 7.90 \text{ ft. per sec.}$$
$$Q_8 = 7.90 \times 0.347 = 2.74 \text{ cu. ft. per sec.}$$

$Q_5 + Q_8 = 1.00 + 2.74 = 3.74$ cu. ft. per sec. which is greater than Q_6. Evidently E_D is too high and must be lowered. Try $E_D = 25.0$. Estimating values of f from the first approximation,

$$f_6 = 0.0200, \qquad f_5 = 0.0231, \qquad f_8 = 0.0203,$$

$$V_6 = \sqrt{\frac{2g \times 75}{0.5 + 19.8}} = 15.4 \text{ ft. per sec.}$$

$$Q_6 = 15.4 \times 0.201 = 3.10 \text{ cu. ft. per sec.}$$

$$V_5 = \sqrt{\frac{2g \times 10}{16.5 + 1}} = 6.07 \text{ ft. per sec.}$$

$$Q_5 = 6.07 \times 0.139 = 0.843 \text{ cu. ft. per sec.}$$

$$V_8 = \sqrt{\frac{2g \times 25}{30.4 + 1}} = 7.17 \text{ ft. per sec.}$$

$$Q_8 = 7.17 \times 0.347 = 2.49 \text{ cu. ft. per sec.}$$

$Q_5 + Q_8 = 0.84 + 2.49 = 3.33$ cu. ft. per sec., while $Q_6 = 3.0$ cu. ft. per sec. E_D is still too high and must be lowered again. Assume $E_D = 22.5$, $f_6 = 0.0199$, $f_5 = 0.0234$, $f_8 = 0.0207$,

$$V_6 = \sqrt{\frac{2g \times 77.5}{0.5 + 19.7}} = 15.7 \text{ ft. per sec.}$$

$$Q_6 = 15.7 \times 0.201 = 3.16 \text{ cu. ft. per sec.}$$

$$V_5 = \sqrt{\frac{2g \times 7.5}{16.7 + 1}} = 5.23 \text{ ft. per sec.}$$

$$Q_5 = 5.23 \times 0.139 = 0.726 \text{ cu. ft. per sec.}$$

$$V_8 = \sqrt{\frac{2g \times 22.5}{31.1 + 1}} = 6.72 \text{ ft. per sec.}$$

$$Q_8 = 6.72 \times 0.347 = 2.33 \text{ cu. ft. per sec.}$$

$$Q_5 + Q_8 = 0.73 + 2.33 = 3.06 \text{ cu. ft. per sec.}$$

$$Q_6 = 3.16 \text{ cu. ft. per sec.}$$

This is close to a correct result. Closer approximations to the correct value of E_D can be made by computing the energy loss in the 6-in. line using the last approximate value of V_6. The process may be continued until the final results agree as closely as desired, but greater precision is not warranted because the shock losses at the division point have not been considered.

Exact analysis of the problem would require experimental information on the losses at the division and these depend on the nature of the division, whether a Y-branch or one of several types of tees. As an approximation to these losses, one can apply the appropriate equations for loss at bends, sudden contractions, or abrupt expansions, as conditions require. In this case, there is a reduced velocity in each branch and the equation for shock loss applies approximately. Using the last approximation,

$$h_{s6\text{-}5} = \frac{(15.7 - 5.2)^2}{64.4} = 1.71 \text{ ft.}$$

$$h_{s6\text{-}8} = \frac{(15.7 - 6.7)^2}{64.4} = 1.26 \text{ ft.}$$

Since the flow is proportional to the square root of the difference in head, the discharge through the two lines would be reduced in approximately the ratio 0.88 and 0.97, respectively, thus increasing the difference between $Q_5 + Q_8$ and Q_6 of the last approximation. A greater value of E_D should be assumed and the process repeated.

In the event that the velocity heads and the losses at the division point are appreciable as compared with the friction losses, the graphical method is about as tedious as the algebraic.

104. Cavitation.—In a number of important hydraulic problems, the conditions may be such that the absolute pressure at some point in the flow system is reduced to the vapor pressure of the liquid flowing. A further reduction in pressure is impossible, and the rate of discharge is determined by this condition rather than by the over-all head applied. At such points, the liquid breaks apart to form cavities and the phenomenon is known as *cavitation.* It is frequently observed in the operation of pumps and is the cause of the familiar "cut-off" point at high rates of flow. The occurrence of cavitation not only limits the discharge but may also reduce the efficiency, and leads to a number of mechanical difficulties so that careful examination of every flow system for points of excessively low pressure should be a routine procedure.

Consider the general problem of a liquid initially under a total head

$$h = \left(\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1\right)$$

The pressure at any downstream point, at which the elevation is z_2, is $\frac{p_2}{w} = h - z_2 - (1 + K_2)\frac{V_2^2}{2g}$, where K_2 is an over-all loss coefficient for the line between points (1) and (2) based on the velocity V_2. The pressures may be expressed as absolute or gage provided that all are referred to the same datum. If the pressure p_2 equals p_v, the vapor pressure of the liquid at the existing temperature, the maximum value of V_2 is

$$V_{2\max} = \sqrt{\frac{2g\left(h - z_2 - \frac{p_v}{w}\right)}{1 + K_2}}$$

Applying this equation to the problem of flow in the suction line

of a pump which is supplied from a large reservoir in contact with the atmosphere,

$$h = \frac{p_a}{w}$$

$$V_{2\max} = \sqrt{\frac{2g\left(\frac{p_a}{w} - \frac{p_v}{w} - z_2\right)}{1 + K_2}}$$

Here, z_2 is the elevation of the point considered above the surface of the supply sump and would be chosen for the points of both greatest elevation and greatest velocity, if these two points do not coincide.

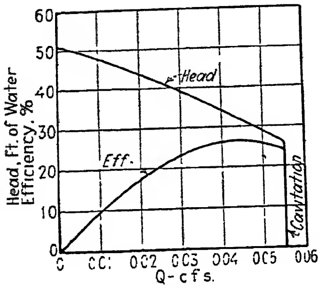

FIG. 114.—Characteristic curves of a small jet pump showing effect of cavitation.

If the conditions are such as to cause cavitation simultaneously over a cross section, the break in the discharge curve is abrupt but with local variations in velocity or elevation, cavitation may develop gradually with a corresponding effect on the discharge curve. Figure 114 shows the effect of cavitation on the discharge of a small jet pump placed with its axis vertical. The cutoff is abrupt. Similar curves for centrifugal pumps show a more gradual transition between the normal head-capacity curve and the cavitation condition.

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Problems

1. A discharge line from a pump has 1000 ft. of 6-in. pipe, six standard elbows, two globe valves, and one gate valve, and discharges submerged into a reservoir. The pump intake is 20 ft. of 6-in. pipe and the suction manometer (air-mercury) differential reading is 20.0 in. A discharge pressure gage located 3 ft. above the suction manometer connection registers 170 lb. per sq. in. If the quantity pumped is 4 cu. ft. per sec., what is the maximum height through which the water can be raised? Assume the pump provides the necessary head. *Ans.* 97.1 ft.

2. A centrifugal pump draws water from a point 2 ft. below the surface of a reservoir through 5 ft. of 4-in. pipe. It discharges through 90 ft. of 3-in. pipe laid horizontally, an elbow, a gate valve and 20 ft. of vertical 3-in. pipe, another elbow, and discharges at the end of 10 ft. of 3-in. pipe at a point 20 ft. above the intake. The pipe is new wrought iron. The velocity in the 3-in. pipe is 20 ft. per sec. Compute the head which must be supplied by the pump considering all losses.

3. A fluid flows from a reservoir through 100 ft. of 4-in., and 60 ft. of 3-in. old cast-iron pipe, and discharges at a point 40 ft. below the surface of the reservoir. Find the rate of discharge if the fluid is (*a*) water, (*b*) Zerolene No. 3 at 90 deg. Fahr.

Ans. (*a*) 0.858 cu. ft. per sec.; (*b*) 0.61 cu. ft. per sec.

4. Two reservoirs whose surface levels are 30 ft. different in elevation are connected by 100 ft. of 3-in. pipe, 50 ft. of 6-in. pipe, and 20 ft. of 4-in. pipe. Coefficients of entrance, expansion, and contraction loss may be taken at 0.03, 0.07, and 0.04, respectively. Assume the pipe is new wrought iron, compute the discharge and plot to scale the energy and hydraulic gradients.

5. A reservoir discharges through 1000 ft. of 6-in. new wrought-iron pipe terminated by a 3-in. nozzle. The difference in elevation between the water surface in the reservoir and the center of the nozzle is 120 ft. Compute the velocity and quantity of discharge and the pressure behind the nozzle. The coefficient of entrance loss to the pipeline is 0.5 and the coefficient of energy loss in the nozzle is 0.04.

6. Compute the diameter of wrought-iron pipe necessary for discharging 1500 gal. per min. of water, the pipe being 1000 ft. long, and its discharging end 4 ft. lower than the surface of the reservoir supplying it. Neglect entrance loss.

7. A pipeline 1000 ft. long discharges in the atmosphere 100 ft. below the surface of a supply reservoir. If the line is to discharge 2.0 cu. ft. per sec., what size of new galvanized pipe must be used? Neglect entrance loss.

8. Two cast-iron pipelines in fair condition are connected to a reservoir and have free discharge at the outlet ends. The first is a 6-in. diameter pipe 1000 ft. long with outlet 12.2 ft. below the reservoir water level. The second is an 8-in. diameter pipe 2000 ft. long. The combined discharge of the two lines is 1.95 cu. ft. per sec. Neglecting entrance losses, determine the elevation of the outlet of the second pipeline with respect to the reservoir water level. *Ans.* 12.1 ft.

9. Water flows from a tank through 300 ft. of 2-in. galvanized wrought-iron pipe at the end of which is a $\frac{1}{2}$-in. nozzle. The total head is 200 ft. The nozzle coefficient is 0.05. What is (*a*) the velocity of the jet? (*b*) the discharge? (*c*) the horsepower of the jet?
Ans. (*a*) 110 ft. per sec.; (*b*) 0.150 cu. ft. per sec.; (*c*) 3.18 hp.

10. A small penstock leading from a reservoir to an impulse water turbine is composed of 5000 ft. of 12-in. steel pipe and 2000 ft. of 8-in. pipe. At the end of the 8-in. pipe is a 2-in. diameter nozzle whose coefficient is 0.07. If the elevation of the nozzle is 1600 ft. below the water surface in the reservoir, compute (*a*) the discharge in cubic feet per second; (*b*) the horsepower of the jet. *Ans.* (*a*) 6.08 cu. ft. per sec.; (*b*) 830 hp.

11. A 12-in. wrought-iron pipeline in good condition discharges water 300 ft. below the surface of a reservoir. Determine the diameter of nozzle which will deliver the maximum power. *Ans.* 4.06 in.

12. A 36-in. diameter pipeline carrying 50 cu. ft. per sec. of water is to be replaced by three pipes which will give the same head loss for the same total discharge. Taking d_1 as the diameter of one pipe, the diameters of the other two are $2d_1$ and $3d_1$. Determine d_1. *Ans.* 10.6 in.

13. Two steel pipes, one 8 in. in diameter and 600 ft. long, and the other 6 in. in diameter and 200 ft. long, lead from separate reservoirs having the same surface elevation. The pipes unite at a branch point from which a 12-in. pipe 800 ft. long leads to a water tank whose water surface is 50 ft. below that of the reservoirs. There is a standard screw gate valve and a 90-deg. elbow in each line. Considering all losses, compute the rate and direction of flow in each pipe.
Ans. From both reservoirs to the water tank; 8-in. line, 3.45 cu. ft. per sec.; 6-in. line 2.75 cu. ft. per sec.; 12-in. line 6.20 cu. ft. per sec.

CHAPTER VIII

UNSTEADY FLOW IN PIPELINES

The problems in the preceding chapters have for the most part assumed steady flow, *i.e.*, the average velocity and pressure at each point were assumed not to vary with time. However, the initiation of such steady conditions involves a transient or "unsteady" period and it is the purpose of this chapter to present some of the basic principles of unsteady flow as they apply to pipelines. Although these principles are in themselves relatively simple, application generally leads to a complex set of equations and, for this reason, only the most elementary problems will be considered.

INCOMPRESSIBLE FLUIDS

The first problems to be considered will be those for which it is permissible to assume that the fluid and the pipe are inelastic. This condition is equivalent to the assumption that the velocity of sound is infinite and that pressure changes are propagated throughout the system in zero time. The practical equivalent of this assumption is merely that the changes in velocity occur in time intervals which are long as compared with the time for a pressure wave to traverse the system. Therefore, it is assumed that the density does not change appreciably and that the liquid filling a straight section of pipe behaves as a solid plug as regards its inertial effects.

The methods employed are suitable for pipelines, *i.e.*, for flow systems which are long in comparison with the diameter. The procedure is to divide the pipeline into sections of constant diameter, in which inertia and friction are effective, joined by transition sections having negligible inertia. In Chap. III it was shown that the basic equation of flow at constant density is

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{\partial z}{\partial s} + \left(\frac{\partial V}{\partial t}\right)_s + \frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)_t = 0 \qquad (3.4)$$

Here, s indicates the direction of flow.

105. Unsteady Flow in a Pipe of Uniform Diameter.—Equation (3.4) does not include the frictional resistance which always acts in a direction opposite to that of the velocity. From the previous discussion of resistance in pipes, it is evident that the resistance term can be written as

$$\frac{f}{D}\frac{V^2}{2}\,ds$$

The differential equation for this condition is then

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)ds + g\left(\frac{\partial z}{\partial s}\right)ds + \left(\frac{\partial V}{\partial t}\right)_s ds + \frac{\partial}{\partial s}\left(\frac{V^2}{2}\right)ds + \frac{f}{D}\frac{V^2}{2}\,ds = 0$$

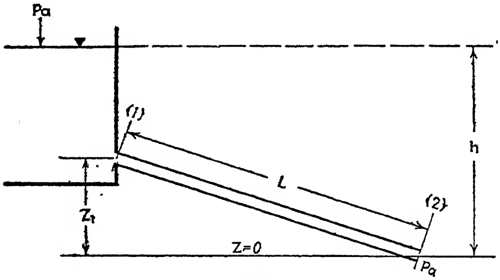

FIG. 115.

For a constant diameter,

$$\frac{\partial}{\partial s}(V^2) = 0$$

Integrating over a length L, and dividing through by g, the equation becomes

$$\frac{\Delta p}{w} + \Delta z + \frac{L}{g}\frac{dV}{dt} + \frac{fL}{D}\frac{V^2}{2g} = 0 \tag{8.1}$$

To illustrate the significance of Eq. (8.1) it will be applied to the situation shown in Fig. 115 in which a pipe of constant diameter and length L is connected to a reservoir. The pipe is initially closed at point (2) and the surface of the reservoir is under atmospheric pressure. The lower end of the pipe is completely opened abruptly and the problem is to predict the subsequent behavior of the water in the pipe. This problem, as stated, violates the general assumptions in that abrupt open-

ing of the valve at (2) would in nature generate an acoustic wave of magnitude h but this wave of pressure and velocity would be superimposed on the flow conditions given by the equations under discussion. Furthermore, there are conditions occurring in practice under which the pipeline at (2) would be under atmospheric pressure at zero velocity. This situation occurs in the drive line of the hydraulic ram immediately after the waste valve opens.

If the length L is large as compared with the diameter of the pipe, Eq. (8.1) becomes

$$\frac{p_a - p_1}{w} + z_2 - z_1 + \frac{L}{g}\frac{dV}{dt} + \frac{fL}{D}\frac{V^2}{2g} = 0 \tag{8.2}$$

The point (1) is located just inside the pipeline. From continuity,

$$A_0 V_0 = A_1 V_1, \qquad \frac{dV_0}{dt} = \frac{A_1}{A_0}\frac{dV_1}{dt}$$

where A_0 and V_0 indicate the area and velocity at points to the left of (1). The area A_0 occupied by active fluid increases rapidly, the acceleration decreases and the inertia effect of the fluid to the left of (1) is negligible. This is true in spite of the fact that all of the liquid in the reservoir is being accelerated, because Eq. (8.2) applies to each pound of the liquid. Between any point in the reservoir and point (1), the equation is the same as for steady flow:

$$\frac{p_1 - p_a}{w} + z_1 - h + \frac{V^2}{2g} + K\frac{V^2}{2g} = 0 \tag{8.3}$$

Adding Eqs. (8.2) and (8.3) to obtain the equation for the flow from the reservoir to the point (2) and remembering that $z_2 = 0$,

$$h - \left(1 + K + \frac{fL}{D}\right)\frac{V^2}{2g} = \frac{L}{g}\frac{dV}{dt} \tag{8.4}$$

The next problem is the proper choice of the coefficients f and K. It has been shown previously that they are functions of the Reynolds number. In the present problem of a flow starting from rest, the Reynolds number increases from zero to a terminal value. Values of f and K corresponding to the velocity of steady flow under the head h will be smaller than the average values

during the acceleration but there is a distinct advantage in using the terminal values. The error involved is not great.

Another question is whether or not friction coefficients obtained during steady flow apply to the same velocity during unsteady flow. No information on this point is available and in the absence of it, it appears to be reasonably correct to use values of f and K applicable to the terminal velocity as stated above. The error results in a computed acceleration in excess of the true value.

For steady flow under the head h, $dV/dt = 0$, and

$$h = \left(1 + K_0 + \frac{f_0 L}{D}\right)\frac{V_0^2}{2g} \tag{8.5}$$

Since K_0 and f_0 are to be used, the quantity $\left(1 + K_0 + \frac{f_0 L}{D}\right)$ from Eq. (8.5) may be inserted in Eq. (8.4) giving,

$$h\left[1 - \left(\frac{V}{V_0}\right)^2\right] = \frac{L}{g}\frac{dV}{dt}$$

Separating variables and integrating,

$$t = \frac{LV_0}{gh}\tanh^{-1}\frac{V}{V_0} + C \tag{8.6}$$

The value of the constant is zero for the conditions of this problem ($V = 0$ when $t = 0$).

A different constant of integration would have been found if other initial conditions had been assumed.

Example: A wrought-iron pipe in fair condition has a sharp-edged inlet. Its diameter is 1 in., length 150 ft. and operating head 40 ft. How long after the sudden opening of a valve at the lower end will it take for the velocity in the pipe to reach a value equal to 0.9 the velocity at steady flow?

For steady conditions, assuming $f = 0.0250$,

$$V_0 = \sqrt{\frac{2gh}{1 + C_e + \frac{fL}{D}}} = \sqrt{\frac{2g \times 40}{1 + 0.5 + 0.0250 \times \frac{150 \times 12}{1.048}}}$$

$$= \sqrt{\frac{64.4 \times 40}{1.5 + 43.0}} = 7.61 \text{ ft. per sec.}$$

The correct value of f is 0.0249 which does not change the velocity appreciably.

$$t_{0.9} = \frac{LV_0}{gh} \tanh^{-1} \frac{V}{V_0}, \qquad \frac{V}{V_0} = 0.9, \qquad \tanh^{-1} 0.9 = 1.472$$
$$= \frac{150 \times 7.61 \times 1.472}{32.2 \times 40} = 1.30 \text{ sec.}$$

106. Variable Diameter.—Unsteady flow in a pipeline containing several sections of different diameter may be treated by reducing the line to an equivalent pipeline of constant diameter and applying the equations of the preceding section. The lengths $L_1, L_2, L_3 \ldots L_n$ are large in comparison both with

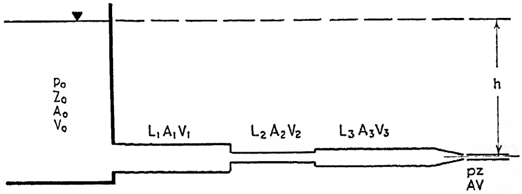

FIG. 116.

the diameters and with the length of the transition section between diameters. Applying Eq. (8.2) to each section of pipe,

$$\frac{\Delta p_1}{w} + \Delta z_1 + \frac{L_1}{g}\frac{dV_1}{dt} + K_1\frac{V_1^2}{2g} = 0$$
$$\frac{\Delta p_2}{w} + \Delta z_2 + \frac{L_2}{g}\frac{dV_2}{dt} + K_2\frac{V_2^2}{2g} = 0$$
$$\frac{\Delta p_3}{w} + \Delta z_3 + \frac{L_3}{g}\frac{dV_3}{dt} + K_3\frac{V_3^2}{2g} = 0$$
$$\frac{\Delta p_n}{w} + \Delta z_n + \frac{L_n}{g}\frac{dV_n}{dt} + K_n\frac{V_n^2}{2g} = 0$$

Here, the K's represent friction and shock losses

$$K = \left(K' + K'' + \frac{fL}{D}\right)$$

For the transitions, which are assumed to have negligible inertia. the pertinent equations are

$$\frac{p_1 - p_0}{w} + z_1 - z_0 + \frac{V_1^2}{2g} + K_1\frac{V_1^2}{2g} - \frac{V_0^2}{2g} = 0$$
$$\frac{p_2 - p_1}{w} + z_2 - z_1 + \frac{V_2^2 - V_1^2}{2g} + K_2\frac{V_2^2}{2g} = 0$$

$$\frac{p_3 - p_2}{w} + z_3 - z_2 + \frac{V_3^2 - V_2^2}{2g} + K_3\frac{V_3^2}{2g} = 0$$

$$\frac{p_n - p_3}{w} + z_n - z_3 + \frac{V_n^2 - V_3^2}{2g} + K_n\frac{V_n^2}{2g} = 0$$

$$\frac{p - p_n}{w} + (z - z_n) + \frac{V^2 - V_n^2}{2g} + \frac{V^2}{2g} = 0$$

Adding equations to obtain the total changes in pressure, elevation, and velocity *at any instant*,

$$\frac{p - p_0}{w} + (z - z_0) + \left(\frac{L_1}{g}\frac{dV_1}{dt} + \frac{L_2}{g}\frac{dV_2}{dt} + \frac{L_3}{g}\frac{dV_3}{dt} + \cdots + \frac{L_n}{g}\frac{dV_n}{dt}\right) + \frac{V^2 - V_0^2}{2g} + \left(K_1\frac{V_1^2}{2g} + K_2\frac{V_2^2}{2g} + K_3\frac{V_3^2}{2g} + \cdots + K_n\frac{V_n^2}{2g} + K\frac{V^2}{2g}\right) = 0 \quad (8.7)$$

For the conditions shown in Fig. 116, $V_0 = 0$, and

$$\left(\frac{p - p_0}{w} + z - z_0\right) = -h,$$

which gives an equation identical in general form with Eq. (8.4).

The equation of continuity for incompressible flow is

$$Q = A_1V_1 = A_2V_2 = \cdots = A_nV_n = AV$$

Differentiating with respect to time at any position,

$$A_1\frac{dV_1}{dt} = A_2\frac{dV_2}{dt} = \cdots = A_n\frac{dV_n}{dt} = A\frac{dV}{dt}$$

Substituting for the velocities and accelerations in Eq. (8.7),

$$\left(\frac{p - p_0}{w}\right) + (z - z_0) + \frac{1}{g}\left(L_1\frac{A}{A_1} + L_2\frac{A}{A_2} + \cdots + L_n\frac{A}{A_n}\right)\frac{dV}{dt} + \frac{V^2}{2g}\left[1 - \left(\frac{A}{A_0}\right)^2 + K_1\left(\frac{A}{A_1}\right)^2 + K_2\left(\frac{A}{A_2}\right)^2 + \cdots + K_n\left(\frac{A}{A_n}\right)^2\right] = 0$$

The area A corresponds to the velocity V. The equivalent length of pipe of area A having the same inertial characteristics as the actual pipeline is

$$L' = \left(L_1\frac{A}{A_1} + L_2\frac{A}{A_2} + \cdots + L_n\frac{A}{A_n}\right)$$

The equivalent head-loss coefficient based on the velocity V is

$$K' = \left[1 - \left(\frac{A}{A_0}\right)^2 + K_1\left(\frac{A}{A_1}\right)^2 + K_2\left(\frac{A}{A_2}\right)^2 + \cdots + K_n\left(\frac{A}{A_n}\right)^2\right]$$

The drop in the hydraulic grade line is

$$-h = \left(\frac{p - p_0}{w}\right) + (z - z_0)$$

and the simplified equation becomes

$$h = \frac{L'}{g}\frac{dV}{dt} + K'\frac{V^2}{2g} \tag{8.8}$$

Integration of Eq. (8.8) leads to Eq. (8.6). It is to be noted that V may be any real or fictitious velocity, provided only that L' and K' are properly computed.

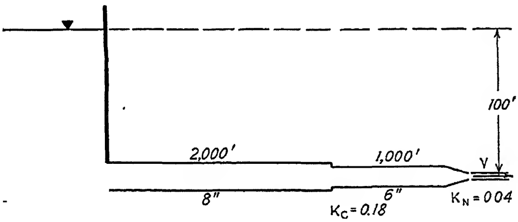

FIG. 117.

The effective inertial length of sections of variable diameter may be approximated as

$$L' = A\int_0^L \frac{dL_1}{A_1}$$

where L_1 is the length along the center line and A_1 is the area of the stream. If the stream does not fill the pipe, it is difficult to select proper values for A_1.

Example: If a 2-in. nozzle at the end of the 6-in. line of Fig. 117 be suddenly opened, what will be the velocity of the jet at the end of 1, 3, 5, 10, 20 seconds?

For steady-flow conditions,

$$100 = 0.5\frac{V_8^2}{2g} + \frac{f_8L_8}{D_8}\frac{V_8^2}{2g} + 0.18\frac{V_6^2}{2g} + \frac{f_6L_6}{D_6}\frac{V_6^2}{2g} + 0.04\frac{V^2}{2g} + \frac{V^2}{2g}$$

$$V = \sqrt{\frac{2g \times 100}{0.00198 + 11.88f_8 + 0.00214 + 23.5f_6 + 1.04}}$$

$$= 59.2 \text{ ft. per sec.}$$

$$L' = L_8\left(\frac{A}{A_8}\right) + L_6\left(\frac{A}{A_6}\right) = 2000\left(\frac{0.0218}{0.347}\right) + 1000\left(\frac{0.0218}{0.201}\right)$$
$$= 125.5 + 108.5 = 234 \text{ ft.}$$

Since the flow starts from zero velocity, C of Eq. (8.6) is zero, and the equation can be rewritten

$$\frac{V}{V_0} = \tanh \frac{gHt}{LV_0}, \qquad V = V_0 \tanh \frac{gHt}{LV_0}$$

Inserting the proper values, and noting that L' should be used for L,

$$V = 59.2 \tanh \frac{32.2 \times 100}{234 \times 59.2}t$$
$$= 59.2 \tanh 0.232t$$

Time t, sec.	$0.232t$	$\tanh 0.232t$	V
0	0	0	0
1	0.232	0.228	13.5
3	0.696	0.601	35.6
5	1.160	0.821	48.6
10	2.32	0.981	58.1
20	4.64	1.000	59.2

107. Variable Head.—In a number of important problems, notably that of the surge chamber, the accelerating or retarding head is itself variable with time. For *incompressible* fluids, the equations previously derived for unsteady flow apply to variable as well as constant heads. For real fluids, the equations are valid provided that the changes in head occur during time intervals which are long as compared with the time required for a pressure change to traverse the system.

Referring to Fig. 118, the flow in the pipeline is controlled by a valve at (2) and is connected to a large chamber at (1), known as a *surge chamber*. The purpose of the surge chamber is to eliminate the excessive variations in pressure which would otherwise result from rapid changes in the setting of the valve (2). As applied to the control of reaction water turbines, the functions of the surge chamber are the following:

1. During closure of the turbine gates, the surge chamber takes the difference between the discharge required by the turbine and that supplied by the penstock. It provides a *retarding* head which gradually equalizes the flow.

2. During opening movements, the surge chamber makes up the deficiency of flow from the penstock and, as it is drawn down, provides an *accelerating* head which increases the rate of flow to that required by the turbines. Surge chambers are also used to prevent excessive pressure fluctuations in long pump lines. In general, surge chambers or other pressure-control devices are usually provided when $T < 20L/c$ where T is the minimum closing time of the valve, L is the length of the line, and c is the velocity of a pressure wave in the pipe.

As an example of the solution of problems on variable head, it will be assumed that, for the arrangement of Fig. 118, the valve at (2) has been set so as to obtain an initial steady velocity V_0 in the pipeline and an elevation z_0 in the surge chamber. The

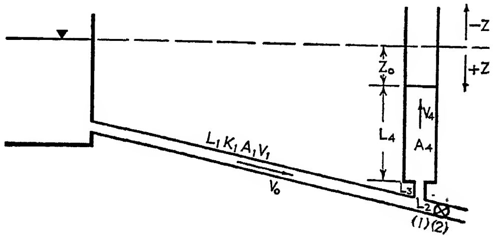

FIG. 118.

valve is then closed abruptly and completely and the problem is to find the maximum height attained by the water surface in the standpipe and the corresponding time. To simplify the problem, the following assumptions will be made:

a. The lengths L_2 and L_3 are negligible as compared with L_1.

b. The area of the surge chamber, A_4, is sufficiently large in comparison with A_1 that the quantity $L_4 \dfrac{dV_4}{dt}$ is negligible as compared with $L_1 \dfrac{dV_1}{dt}$.

c. The pressure head at (1) equals the vertical distance below the instantaneous water level in the surge chamber.

Under these conditions, the equation is

$$z - K_1\frac{V_1^2}{2g} = \frac{L_1}{g}\frac{dV_1}{dt} \tag{8.9}$$

Here, L_1 and K_1 may be regarded as reduced values representing any system of piping between the reservoir and the point (1). From continuity,

$$A_1V_1 = A_4V_4 = -A_4\frac{dz}{dt}$$

after valve (2) is closed. The acceleration in the pipeline can therefore be expressed as

$$\frac{dV_1}{dt} = \frac{dV_1}{dz}\frac{dz}{dt} = -\frac{A_1}{A_4}V_1\frac{dV_1}{dz}$$

Substituting in Eq. (8.9),

$$z + \frac{L_1}{g}\frac{A_1}{A_4}\frac{V_1\,dV_1}{dz} - \frac{K_1V_1^2}{2g} = 0$$

Inserting the following quantities,

$$q = V_1^2, \qquad m = \frac{L}{2g}\frac{A_1}{A_4}, \qquad n = \frac{K_1}{2g}$$

the equation reduces to

$$\frac{dq}{dz} - \frac{n}{m}q + \frac{z}{m} = 0$$

The solution of this equation is

$$q = V_1^2 = \frac{z}{n} + \frac{m}{n^2} + Ce^{\frac{n}{m}z} \tag{8.10}$$

where C is a constant of integration. The initial conditions are $t = 0$, $V_1 = V_0$, $z = z_0$.

$$C = \left(V_0^2 - \frac{z_0}{n} - \frac{m}{n^2}\right)e^{-\frac{n}{m}z_0}$$

Inserting this value of the constant in Eq. (8.10) and solving for $z = z_{max}$ at $V_1 = 0$ gives as the equation for z_{max}

$$\frac{z_{max}}{n} + Ce^{\frac{n}{m}z_{max}} + \frac{m}{n^2} = 0 \tag{8.11}$$

which is best solved by trial and error neglecting the middle term as a first approximation. The elapsed time between

closure of the valve and elevation z_{max} is obtained from Eq. (8.10):

$$V_1 = -\frac{A_4}{A_1}\frac{dz}{dt} = \sqrt{\frac{z}{n} + \frac{m}{n^2} + Ce^{\frac{n}{m}z}}$$
$$dt = -\frac{A_4}{A_1}\frac{dz}{\sqrt{\frac{z}{n} + \frac{m}{n^2} + Ce^{\frac{n}{m}z}}}$$

This equation can be integrated graphically by plotting

$$\left(-\frac{A_4}{A_1}\frac{1}{\sqrt{\frac{z}{n} + \frac{m}{n^2} + Ce^{\frac{n}{m}z}}}\right)$$

against z and planimetering the area under the curve between z_0 and z_{max}.

In spite of the fact that the problem under consideration was extremely simple, integration of the equation of motion involved some difficulties. Considerably more trouble would have been encountered if the valve (2) had been located some distance from the branch to the surge chamber and had been closed gradually. For such conditions the method of finite increments appears to be more suitable than integration of the equations because the latter requires restrictive assumptions. For the simple problem treated above the method of finite increments would be as follows:

$$z - K_1\frac{V_1^2}{2g} = \frac{L_1}{g}\frac{dV_1}{dt}$$
$$V_1 = -\frac{A_4}{A_1}\frac{dz}{dt}$$

Rewriting and substituting finite increments for the differentials,

$$\Delta V_1 = \frac{g}{L_1}\left(z_m - K_1\frac{V_{1m}^2}{2g}\right)\Delta t$$
$$\Delta z = -V_1\frac{A_1}{A_4}\Delta t$$

Here, z_m and V_{1m} are mean values of z and V_1 during a short time interval Δt_1. Using the values obtained as a first approximation, the solution is repeated using

$$z_m' = z_0 + \frac{\Delta z}{2}$$

$$V_{1m}' = V_1 + \frac{\Delta V}{2}$$

until consistent values are obtained. The process is then repeated for another increment of time Δt_2, using the values of V_1 and z obtained for the end of the interval Δt_1. This step-by-step method has the advantage of permitting use of a friction coefficient variable with velocity and of giving any desired degree of accuracy in the computation.

EFFECT OF COMPRESSIBILITY

Since all fluids including water are compressible and show a finite velocity of wave propagation in pipes, it follows that the preceding equations for unsteady flow are not entirely correct since any change of head, no matter how slowly it occurs, can be broken up into component changes, each one of which occurs in a period equal to $2L/c$. In other words a change of regime is always accompanied by acoustic phenomena which result in variations in pressure and velocity above and below those indicated by the equations for incompressible flow. The effect is not exactly one of superposition because the friction term is proportional to the square of the velocity but it is very nearly so.

The preceding equations derived for incompressible fluids apply to gases with the same restrictions as for liquids, plus the requirement that the total change in pressure be small in comparison with the absolute pressure.

108. Water Hammer.—Finite pressure waves in pipelines carrying liquids are referred to as *water hammer* because of the hammering sound which generally accompanies this phenomenon. The theory of water hammer has been treated in considerable detail by Joukowsky, Allievi, Dahl, and others. A very good summary of theoretical and experimental work on water hammer has recently been published by the American Society of Mechanical Engineers.

Every student of elementary physics is familiar with the problem of the motion of acoustic waves in organ pipes. Pressure reflections occur at the open and closed ends and one can compute from the velocity of sound and the length of the pipe, the natural period of vibration, or the pitch of the system. The treatment

usually assumes that the pressure variations are small as compared with the absolute pressure. This analysis can be applied almost without alteration to the phenomenon of water hammer, which deals with finite pressure changes in liquids. Every change of velocity with time is accompanied by a pressure wave which travels back and forth through the pipeline just as does an acoustic wave in an organ pipe.

Derivation of the basic equations for water hammer proceeds from the equations of momentum and continuity. The symbols used in the derivation have the following significance:

A, ΔA = area and change in area.
p, Δp = pressure and change in pressure.
V, ΔV = velocity and change in velocity.
w, Δw = unit weight and change in unit weight of liquid.
c = velocity of sound in pipelines.
Δt = time interval.
ΔT = change in wall tension per unit area.
D = diameter of pipe.
S = thickness of pipe walls.
E = linear modulus of elasticity of pipe material.
l = inner circumference of pipe.
e = circumferential stretching of pipe wall per unit length.
K = bulk modulus of liquid.

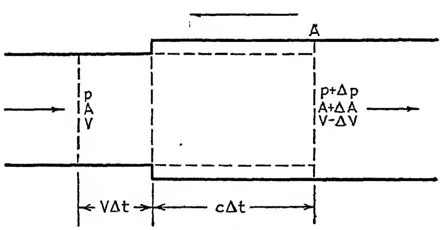

Fig. 119.

Figure 119 represents schematically and to a very exaggerated scale the passage of a water hammer wave at velocity c through a pipeline in which the initial velocity in the opposite direction was V. During progress of this pressure wave along the pipe, not only is the liquid compressed but the pipe is both expanded and elongated. The effect of the elongation is slight for pipes well

anchored at the ends, and as a first approximation we shall consider the expansion alone. The velocity of flow will be assumed to have been changed abruptly by an amount ΔV, and the conditions at the front of the advancing wave will be assumed to approximate those shown where the pressure change caused an abrupt expansion at A.

The mass of liquid experiencing the change in velocity ΔV during the time interval Δt is

$$\frac{Aw}{g}(V\Delta t + c\Delta t)$$

The equation of impulse and change of momentum is

$$A\Delta p\Delta t = \frac{Aw}{g}(V\Delta t + c\Delta t)[V - (V - \Delta V)]$$

$$\Delta p = \frac{w}{g}(V + c)\Delta V$$

The quantity V is usually small as compared with c for liquids and we may write

$$\left.\begin{aligned} \Delta p &= \frac{w}{g}c\Delta V \\ \Delta h &= \frac{c\Delta V}{g} \end{aligned}\right\} \qquad (8.12)$$

In order to obtain c, the velocity of the pressure wave, it is necessary to utilize the continuity relationship. Considering the region in Fig. 119 marked out by the advancing pressure wave in time Δt, the weight of fluid entering the region during this time interval is $AVw\Delta t$ while that leaving is

$$(V - \Delta V)(A + \Delta A)(w + \Delta w)\Delta t$$

The weight rate of storage of fluid in the region consists of two parts, one corresponding to the increase of area and equal to $c(w + \Delta w)\Delta A\Delta t$, and the other amounting to $cA\Delta w\Delta t$ resulting from the increase in specific weight. The complete continuity equation is, therefore,

$$AVw\Delta t = c\Delta A\Delta t(w + \Delta w) + Ac\Delta w\Delta t + (V - \Delta V)(A + \Delta A)(w + \Delta w)\Delta t \qquad (8.13)$$

The quantities ΔA and Δw are not known as yet and must be expressed in terms of the dimensions and elasticity of the pipe and the compressibility of the liquid. From the definition of the bulk modulus of a liquid,

$$\Delta w = \frac{\Delta p\, w}{K}$$

The increase in tension per unit area due to the pressure change Δp is

$$\Delta T = \frac{\Delta p\, D}{2S}$$

and the elongation of the circumference is very nearly

$$\Delta l = \frac{\pi D\, \Delta T}{E} = \frac{\pi D^2 \Delta p}{2SE}$$

The change in area corresponding to a small change in the circumference is

$$\Delta A = \frac{1}{2\pi} l \Delta l = \frac{D}{2} \frac{\pi D^2 \Delta p}{2SE} = \frac{AD}{SE} \Delta p$$

Simplifying Eq. (8.13) and substituting for ΔA and Δw,

$$c + V - \Delta V = \frac{\Delta V}{\Delta p \left(\frac{D}{SE} + \frac{1}{K} + \frac{D}{SE} \frac{\Delta p}{K} \right)}$$

The quantity $V - \Delta V$ is small as compared with c, $\Delta p/K$ is small as compared with unity, and

$$c = \frac{\Delta V}{\Delta p \left(\frac{D}{SE} + \frac{1}{K} \right)} \tag{8.14}$$

But it has been shown previously that [Eq. (8.12)]

$$\Delta p = \frac{w \Delta V c}{g}$$

Substituting this relation in Eq. (8.14) gives

$$c = \sqrt{\frac{g}{w \left(\frac{1}{K} + \frac{D}{SE} \right)}} \tag{8.15}$$

If $E = \infty$, corresponding to a perfectly rigid pipe,

$$c = \sqrt{\frac{Kg}{w}} \tag{8.16}$$

which is the usual expression for the velocity of sound in liquids. Values of E for usual pipe materials are approximately as follows:

Wrought steel.......28×10^6 lb. per sq. in.,
Wrought iron.......26 to 29×10^6 lb. per sq. in.,
Cast iron...........12 to 14×10^6 lb. per sq. in.

For water at 59 deg. Fahr. in a steel pipe with $D/S = 100$, $c = 3280$ ft. per second; while for $D/S = 200$, $c = 2340$ ft. per sec. For $E = \infty$, $c = 4720$ ft. per sec. A similar development by Durand which takes into account the elongation of a pipe fixed at the upper end gives

$$c = \sqrt{\frac{g}{w\left[\frac{1}{K} + \frac{D}{4SE}\left(5 - \frac{4}{\alpha}\right)\right]}}$$

where α is Poisson's ratio for the material of the pipe wall.

109. Reflection of Pressure Waves.—The relationship between change in pressure and change in velocity during water hammer has been found to be linear, which greatly simplifies the treatment of water-hammer problems because any change of regime may be broken up into small component changes which may then be traced back and forth through the piping system provided that damping and reflection of the waves can be predicted. The actual pressure or velocity at any point at any time is the summation of the effect of all component waves at that point and time. The fact that the loss of energy in friction is proportional to the square of the velocity causes a deviation from the results obtained by superposition but this effect appears not to be especially important and is not ordinarily considered.

Reflection of water-hammer waves occurs at every change in area or rigidity as well as at the ends of the pipeline. The reflection coefficients may be obtained as follows: The two conditions which must be fulfilled at a point of change in rigidity or area are that

1. The pressure must be uniform at all times over the cross section at the junction.
2. Continuity relationships must be fulfilled.

If the pressure change of the primary wave is Δp, that of the transmitted wave Δp_t, and that of the reflected wave Δp_r, the equations are

$$\Delta p + \Delta p_r = \Delta p_t \tag{8.18}$$

$$A_r(\Delta V - \Delta V_r) = A_t \Delta V_t \tag{8.19}$$

From Eq. (8.12), the velocity changes corresponding to the pressure changes are

$$\left.\begin{aligned} \Delta V &= \frac{g\Delta p}{wc_r} \\ \Delta V_r &= \frac{g\Delta p_r}{wc_r} \\ \Delta V_t &= \frac{g\Delta p_t}{wc_t} \end{aligned}\right\} \tag{8.20}$$

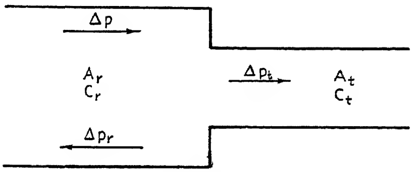

FIG. 120.

Here, c_r and c_t are to be computed from Eq. (8.17). Inserting Eqs. (8.20) in Eq. (8.19) and combining with Eq. (8.18) gives as the magnitude of the reflected and transmitted waves

$$\Delta p_r = -\Delta p \frac{1 - M}{1 + M}$$

$$\Delta p_t = \Delta p \frac{2M}{1 + M}$$

Here, $M = \dfrac{A_r}{A_t}\dfrac{c_t}{c_r}$. The quantity multiplying Δp on the right-hand side is known as the *reflection coefficient.*

Certain special cases are of importance:

Closed End.

$$A_t = 0, \qquad M = \infty$$

$$\frac{\Delta p_r}{\Delta p} = \frac{M - 1}{M + 1} = 1 - 2\left(\frac{1}{M} - \frac{1}{M^2} + \frac{1}{M^3} \cdots\right)$$

$$= +1$$

$\dfrac{\Delta p_t}{\Delta p} = \dfrac{2M}{1 + M} = 2.$ (imaginary wave giving same effect as closed end)

Pressure rises to $2\Delta p$.

Open End.

$$A_t = \infty, \qquad M = 0$$
$$\frac{\Delta p_t}{\Delta p} = 0, \qquad \frac{\Delta p_r}{\Delta p} = -1$$

General.

$$M = 1, \qquad \Delta p_r = 0, \qquad \frac{\Delta p_t}{\Delta p} = 1$$
$$M > 1, \qquad \frac{\Delta p_r}{\Delta p} > 0, \qquad \frac{\Delta p_t}{\Delta p} > 1$$
$$M < 1, \qquad \frac{\Delta p_r}{\Delta p} < 0, \qquad \frac{\Delta p_t}{\Delta p} < 1$$

These coefficients must be applied at every change of area or rigidity for each component wave, whether negative or positive.

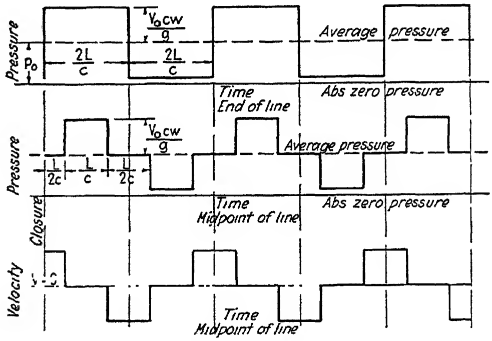

FIG. 121.

110. Pressure Diagrams.—The principal features of water-hammer diagrams are illustrated in Figs. 121 and 122, which have been drawn assuming frictionless flow in a pipeline that is connected to a reservoir at the upper end. The initial velocity of steady flow is V_0. The pressure at the upper end remains constant.

Case I. Instantaneous Closure.—As the pressure wave travels up the pipeline, successive layers of liquid are brought to rest and the pressure rises by an amount Δp. At the time L/c after closure, the pressure wave reaches the upper end of the pipeline and the entire content of the line is at rest. The strain energy then causes a recoil with a velocity, $-V_0$, and the pressure drops

to normal. This effect travels down the pipeline with a velocity c and reaches the closed end at time $2L/c$. The liquid then tends to pull away from the end of the pipe. To bring it to rest, the pressure must drop below the initial pressure by an amount

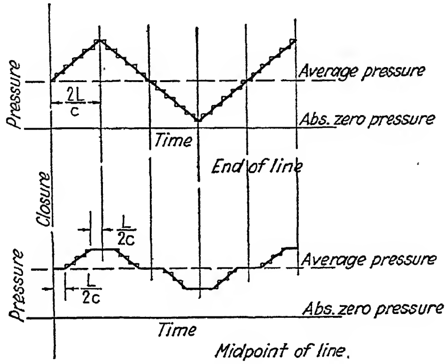

FIG. 122.

Δp ($2\Delta p$ below maximum). The pressure diagrams for points at the lower end of the line and at the middle of its length appear in Fig. 121 where it is assumed that the average initial pressure is greater than Δp. The pressure drops below the initial value

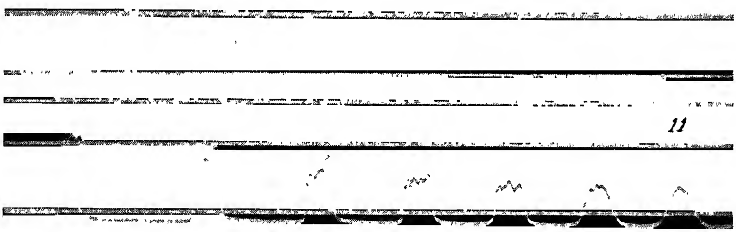

FIG. 123.—Oscillograph record of pressure variations due to water hammer at the end of a pipe line with $p_0 < \Delta p$. Between pressure surges the pressure drops to the vapor pressure and the liquid draws away from the end of the pipe.

by an amount equal to the initial rise, except when the initial absolute pressure p_0 is less than Δp. If $p_0 < \Delta p$, the pressure drops only to the vapor pressure and the diagram becomes asymmetrical with approximately equal areas above and below the average pressure line. Figure 123 shows a pressure diagram measured by a carbon-pile pressure gage with $p_0 < \Delta p$.

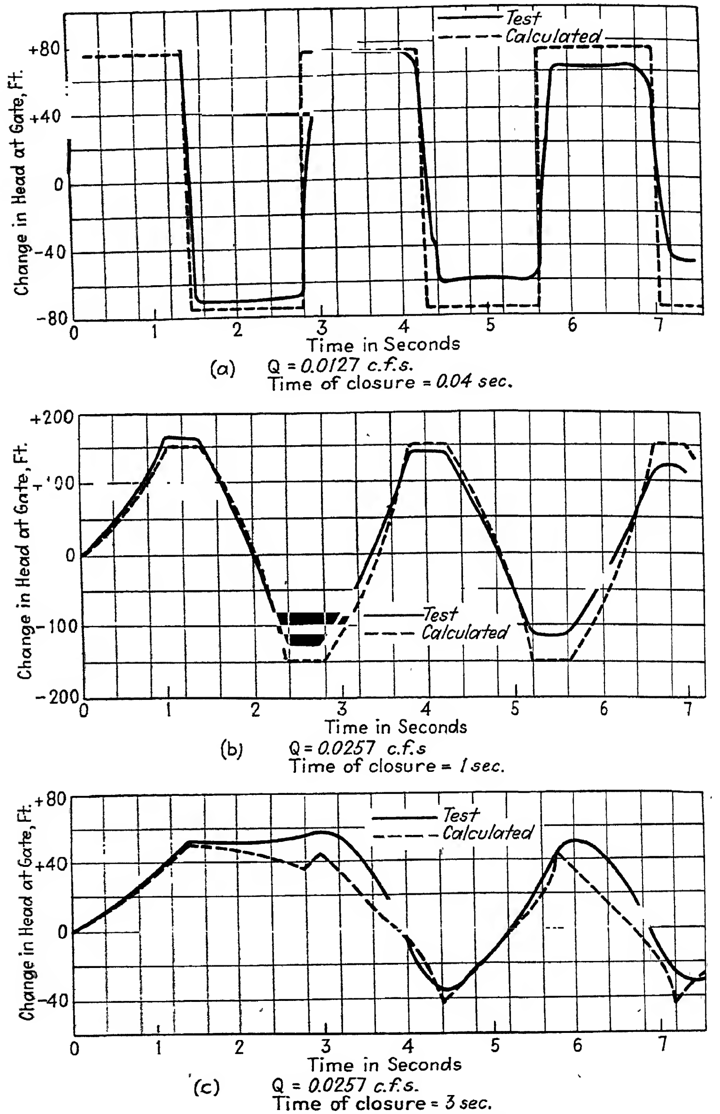

FIG. 124.—Water-hammer studies, comparison of calculated and experimental results, Big Creek Experimental Penstock, Southern California Edison Company. Conditions: $H = 301.6$ ft., $L = 3060$ ft. of 2.06-in. pipe, $2L/c = 1.40$ sec. (*Courtesy Ray S. Quick.*)

Case II. Gradual Closure.—The equations derived apply particularly to small pressure and velocity changes and consequently one may consider small elements of the pressure wave and compute the pressure at any point at any instant by adding the pressures due to the individual elements. In Fig. 122 it has been assumed that the velocity at the lower end is *decreased to zero at a uniform rate* in a time $2L/c$. The rate of gate closure which would bring about a uniform rate of decrease in the line velocity may be determined. The velocity in the line at any valve setting may be expressed as

$$V_n = V_0 \phi_n \sqrt{\frac{H_0 - h_n}{H_0}}$$

where ϕ_n is a coefficient dependent on the valve type and setting and h_n is the summation of the effects of all pressure waves up to the instant considered. If the valve closes in a time less than $2L/c$, no reflected negative waves are involved and

$$h_n = \frac{(V_0 - V_n)c}{g}$$

at any instant during closure.

The effect of friction and impact losses on the magnitude of the pressure wave has not been thoroughly investigated.

111. Experimental Results.—Measurements of water-hammer pressures both in the field and in the laboratory show extremely close agreement, which might even be improved if more reliable pressure-measuring instruments were available. Figure 124 shows a comparison between measured and computed curves for different rates of gate operation. Le Conte has analyzed the phenomenon occurring when $p_0 < \Delta p$ and obtained good agreement between theory and practice. These and other experimental investigations indicate that the phenomenon of water hammer is more amenable to purely theoretical treatment than many apparently less complex problems in fluid flow.

Figure 125 is a generalized diagram prepared by Quick, which gives the pressure rise at the end of a pipeline resulting from uniform gate movement and complete closure in N water-hammer intervals $\left(N = \dfrac{\text{time of closure}}{2L/c}\right)$.

112. Pressure Waves in Gases.—The method used to determine the velocity of a pressure wave in liquids is applicable to gases but some modification of the final equations is necessary to obtain the velocity of advance of a pressure change of appreciable magnitude. The bulk modulus of liquids is nearly constant and of such magnitude that the changes in density and temperature are relatively small and hence Eqs. (8.15), (8.16), and (8.17) apply to pressure waves of all usual magnitudes. However,

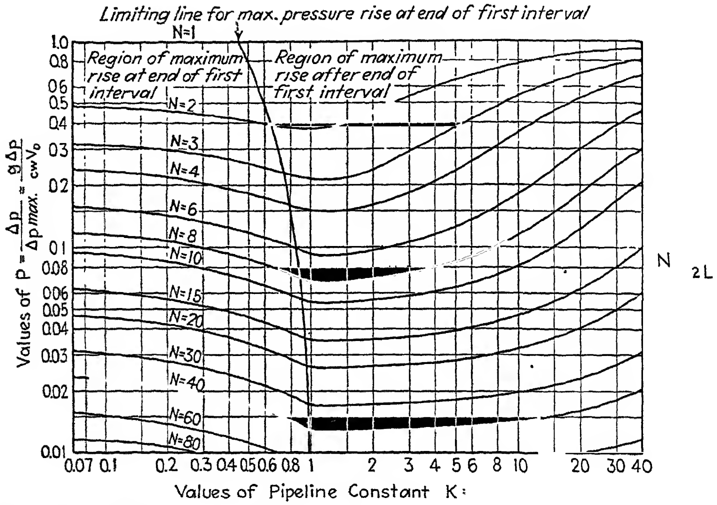

FIG. 125.—Chart showing maximum pressure rise with uniform gate motion and complete closure.

the greater compressibility of gases limits the application of these equations to small pressure changes. The velocity of pressure waves in gases is of basic importance in the theory of acoustics, in which it is usually assumed that the pressure change is small, and as a result the impression appears to be widespread that the velocity of sound is the velocity of propagation of a pressure wave of any magnitude. Equation (8.16) is the usual equation for the velocity of sound in any infinite extent of fluid or in a fluid confined in a perfectly rigid tube. Other forms of this equation involving the gas laws may be obtained by substituting as follows:

$$K = \frac{dp}{\left(\dfrac{d\rho}{\rho}\right)}, \qquad c = \sqrt{\frac{dp}{d\rho}}$$

$$p = g\rho RT, \qquad \frac{p}{\rho^n} = \text{const.}$$

$$\frac{dp}{d\rho} = n\frac{p}{\rho} = ngRT$$

$$c = \sqrt{n\frac{p}{\rho}} = \sqrt{ngRT} \tag{8.21}$$

Here, c is the velocity relative to the fluid, R is the gas constant, n is the exponent representing the expansion, which must be

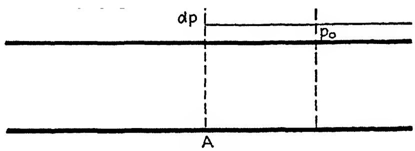

FIG. 126.

very nearly adiabatic, and T is the absolute temperature. The change in velocity of the gas itself resulting from passage of a differential pressure wave may be obtained by considering a rigid tube of uniform area filled with a fluid initially at rest, through which a pressure disturbance of magnitude dp is moving with velocity c (Fig. 126). From the equation for impulse and momentum, the effect of a pressure difference dp, acting during an interval in which the wave is advancing from A to B, is to generate a velocity dV given by the equation

$$\text{Impulse} = \text{change of momentum}$$

$$dp\,dt = (\rho c\,dt)\,dV$$

$$dV = \frac{1}{\rho c}\,dp \tag{8.22}$$

This equation is identical with Eq. (8.12). The convention regarding the signs of p and V is the same as in the theory of water hammer.

The reason for a difference between the velocity of sound and that of a large pressure wave is evident from Eqs. (8.21) and

(8.22). The circumstances which must be considered in treating the problem of large pressure waves are the following:

1. The passage of each elementary pressure wave generates an elementary velocity in the fluid and also increases the density and temperature.

2. A finite pressure wave may be thought of as composed of a succession of elementary waves each one of which travels relative to a medium already set in motion by the preceding elements. In addition, the final elements have a greater relative velocity in a positive wave because the density and temperature have increased.

Substituting from Eq. (8.21) for c in Eq. (8.22),

$$dV = \sqrt{\frac{1}{n\rho p}}\,dp$$

Integrating to obtain the change in the velocity of the fluid resulting from a positive change in pressure equal to $p - p_0$,

$$V_p - V_0 = \frac{2n}{n-1}\sqrt{\frac{p_0^{\frac{1}{n}}}{n\rho_0}}\left[p^{\frac{n-1}{2n}} - p_0^{\frac{n-1}{2n}}\right]$$

If the pressure change takes place rapidly, as is assumed, the compression is adiabatic. In air at 14.7 lb. per sq. in. $\rho_0 = 0.0023$ slug per cu. ft., and $n = 1.41$, the passage of a wave having a maximum absolute pressure of 20 lb. per sq. in. generates a forward velocity of air of 238 ft. per sec.

The discussion thus far has been concerned with the mass velocity generated. It is now time to consider the effect on the wave form of variations in the velocity of sound and the mass velocity. In a single positive surge of pressure, it is evident that the elements of the last wave generated will have the greatest velocity and will overrun the preceding elements, causing the wave front to become steeper until a discontinuity in pressure exists. The whole wave then travels ahead with the velocity of the fastest element.

Indicating by c_p the velocity of a sound wave relative to a fixed point and by c_f the velocity relative to the fluid,

$$c_p = c_f + V_p \tag{8.23}$$

if the fluid was initially at rest ($V_0 = 0$). Substituting from Eq. (8.21) to obtain the velocity of a fully developed positive wave,

$$c_p = \sqrt{gnRT} + \frac{2n}{n-1}\sqrt{\frac{p_0^{\frac{1}{n}}}{\rho_0 n}}\left[p^{\frac{n-1}{2n}} - p_0^{\frac{n-1}{2n}}\right]$$
$$= c_0\left[\left(\frac{p}{p_0}\right)^{\frac{n-1}{2n}}\left(\frac{n+1}{n-1}\right) - \frac{2}{n-1}\right] \tag{8.24}$$

Here, c_0 is the velocity of a small pressure wave (usual velocity of sound) at the original temperature T_0 in the undisturbed state. In air, the velocity of a pressure wave which doubles the existing pressure is 1.6 times as great as the velocity of a small wave at the initial conditions.

The passage of a negative pressure wave (reduction of pressure) can be treated by the same methods but the over-all result is different because each element of the wave generates a fluid velocity which is opposite in direction to that of the wave and the relative velocity of the elements decreases because of the reduced pressure and temperature. Instead of growing steeper the negative wave becomes elongated as it moves through the gas.

Ordinarily the elasticity of the walls of the tube or pipe need not be considered in dealing with pressure waves in gases. However, if the effect of the expansion or elongation is not negligible, the velocity of a finite pressure wave may be obtained by starting from Eq. (8.15) or (8.17) and following the same general procedure as for rigid walls. The rate of damping of finite pressure waves appears not to have been investigated in detail in spite of the fact that it is of importance in a number of engineering problems as, *e.g.*, in the operation of air brakes on trains.

113. Compression Shocks in Expanding Nozzles.—In Chap. III it was shown that the weight rate of discharge of a gas through a convergent nozzle reaches a maximum value at a pressure ratio equal to $\left(\frac{2}{n+1}\right)^{\frac{n}{n-1}}$. By substituting this pressure ratio in Eq. (3.9),

$$V_2 = \sqrt{gnRT_2}$$

Comparison of this result with Eq. (8.21) shows that at the critical pressure ratio the velocity of the gas equals the *local* velocity of sound. In the expanding type of nozzle, such as is

used in steam turbines, the weight discharged reaches its maximum value when the pressure at the narrowest section, or throat, is reduced to this same critical pressure. However, beyond the throat, the velocity of flow may continue to increase if the over-all pressure difference is sufficiently great. If the expansion occurs without shock, the pressure will follow a curve such as p_1-p_2-p_3 in Fig. 127 in which p_2 is the critical pressure corresponding to the initial conditions. The shape of the complete expansion curve is found by using Eq. (3.9) for all points with the rate of discharge W equal to that given by the critical pressure ratio,

$$\frac{p_2}{p_1} = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}}$$

The maximum rate of discharge is

$$W_{\max} = p_1 A_2 \sqrt{\left(\frac{2g}{RT_1}\right)\left(\frac{n}{n+1}\right)\left(\frac{2}{n+1}\right)^{\frac{2}{n-1}}}$$

The corresponding throat velocity (velocity of sound) is

$$V_2 = \sqrt{2g\left(\frac{n}{n+1}\right)RT_1}$$

The relationship between p_3 and p_2 for complete expansion depends on the ratio of the areas. It is assumed that the angle of divergence is small enough that the flow does not separate from the side walls.

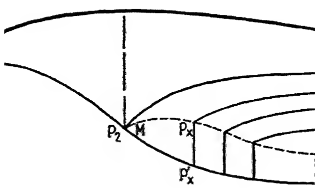

FIG. 127.

Considering the complete expansion curve in Fig. 127, the flow velocity beyond point (2) is greater than the local velocity of sound. This fact is of interest in connection with the velocity of finite pressure waves and, in fact, shows that waves of finite magnitude must travel at a velocity greater than the velocity of sound. For example, if it is assumed that pressure waves of all magnitudes travel with the velocity of sound, then it becomes

impossible to change the flow conditions by increasing the downstream pressure, p_3, because the fluid is moving at a velocity greater than that of the pressure change. Following this line of reasoning, it would be possible to make p_3 greater than p_1 without reducing the discharge, which is evidently absurd since the total energy per unit weight would increase in the direction of flow.

In Fig. 127, the line p_2-p_4 represents the condition of critical pressure at the throat. The distribution of pressure in the nozzle for downstream pressures between p_3 and p_4 has been investigated by Tietjens in the following manner: It is known that under such conditions there is an abrupt increase in pressure, known as a *compression shock*, at some point in the expanding section. This condition may be expected to give rise to a shock loss, and therefore the equation for mechanical energy may not be applied but the total energy and momentum equations are valid. Since the change in pressure occurs in a very short distance, the area may be assumed to be constant. The momentum equation is then

$$A(p_x'' - p_x') = \frac{W}{g}(V_x' - V_x'')$$

where V_x' and p_x' apply to conditions before the compression shock and V_x'' and p_x'' to conditions after it. The subscript x indicates the position in the nozzle. From continuity,

$$W = \frac{AV_x'}{v_x'} = \frac{AV_x''}{v_x''}$$

and

$$p_x'' - p_x' = \frac{V_x'V_x''}{g}\left(\frac{1}{v_x''} - \frac{1}{v_x'}\right)$$

With V_1, $z_1 - z_2$, and ${}_1R_2$ equal to zero, the total-energy equation [Eq. (3.14)] becomes

$$V_x' = \sqrt{\frac{2gn}{n-1}(p_1v_1 - p_x'v_x')}$$

$$V_x'' = \sqrt{\frac{2gn}{n-1}(p_1v_1 - p_x''v_x'')}$$

Dividing through the first equation by $(v_x')^{\frac{1}{2}}$ and the second by

$(v_x'')^{\frac{1}{n}}$, subtracting the second from the first, and solving for the pressure difference,

$$p_x'' - p_x' = p_1v_1\left(\frac{1}{v_x''} - \frac{1}{v_x'}\right) + \frac{n-1}{n}\frac{V_x'V_x''}{2g}\left(\frac{1}{v_x''} - \frac{1}{v_x'}\right)$$

Equating the two expressions for the pressure difference and canceling out common terms,

$$V_x'V_x'' = \frac{2gn}{n+1}p_1v_1 = c_2^2 = \left(\frac{1}{n+1}\right)c_1^2$$

Here c_1 and c_2 are the velocity of sound in the upper chamber and at the throat, respectively. From continuity,

$$\frac{V_x'}{V_x''} = \left(\frac{p_x''}{p_x'}\right)^{\frac{1}{n}}$$

$$\frac{p_x''}{p_x'} = \left[\frac{1}{n+1}\left(\frac{c_1}{V_x'}\right)^2\right]^n$$

The product of the flow velocities before and after the compression shock is equal to the square of the velocity of sound at the critical pressure. At each point in the diverging section there are two velocities of flow which satisfy the total energy and momentum equations, and the flow may change abruptly from the higher velocity to the lower with a corresponding increase of pressure and entropy and a loss of mechanical energy. An abrupt change from the lower to the higher velocity would result in an increase of energy and is impossible. The point at which the shock will occur may be determined by drawing the complete expansion line by means of Eq. (3.9) and then computing for each point x, the velocity V_x'' and the pressure p_x'' which would result if the shock occurred at that point. This procedure gives the dotted line MN in Fig. 127. The next step is to apply Eq. (3.9) to the region beyond the shock, starting back from the pressure p_5. The intersection of the two lines gives the location of the compression shock. It should be noted that the critical pressure may exist at the throat of an expanding nozzle when p_5/p_1 is greater than the critical ratio.

The compression shock remains fixed in position if the external conditions remain constant and therefore it must move upstream relative to the fluid with a velocity, c_f, equal to the fluid velocity

V_x'. The problem might therefore be solved by computing the pressure ratio, (p_x''/p_x'), from Eq. (8.24) for each point which would give a velocity c_f equal to V_x' at that point. Here, c_0 is the velocity of sound at the condition p_x'.

The analysis of compression shocks is very nearly identical with that for the hydraulic jump in an open channel. In both cases, the upstream velocity must exceed the velocity of a small wave and the phenomenon is accompanied by a loss of available mechanical energy. The rise in pressure and increase of density in the compression shock correspond to the increase in depth of the jump. Both phenomena may be treated as waves moving with relative velocities equal and opposite to the absolute velocity of the approaching fluid.

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Problems

1. If the nozzle of Prob. 9, Chap. VII, has been plugged and is opened suddenly, how long will it require for the velocity of the jet to reach 0.95 times the velocity of steady flow? *Ans.* 0.236 sec.

2. A reservoir discharges through 100 ft. of 12-in. pipeline under a 10-ft. head. At the end of the pipeline there is a 3-in. nozzle with a coefficient of 0.05. If the nozzle is suddenly opened, what is the velocity of discharge at the end of the first 10 sec. and how long will it require for the velocity of discharge to become 0.2 ft. per sec. less than its final value? Assume f constant at 0.024. *Ans.* 1.29 sec.

3. Two water tanks are connected by a pipeline composed of 1000 ft. of 12-in. and 500 ft. of 8-in. pipe. If the difference in water level in the two tanks is 20 ft. and the smaller pipe empties into the lower tank, how long is required for the discharge to become one-half the terminal value if a valve at the discharge end is suddenly opened? (Take values of f from Table 13.) *Ans.* 6.22 sec.

4. Water discharges from a reservoir through a 2-in. nozzle at the end of 500 ft. of 12-in. and 500 ft. of 8-in. wrought-iron pipe. The nozzle is 30 ft. below the surface of the water in the reservoir. Assume entrance, contraction, and nozzle loss coefficients as 0.5, 0.05, and 0.04, respectively. If the nozzle is suddenly opened, what will be (*a*) the velocity of the jet at the end of 5 sec.? (*b*) the total discharge during the first 5 sec.?
Ans. (*a*) 41.1 ft. per sec.; (*b*) 3.3 cu. ft.

5. A conduit 7 ft. in diameter and 37,000 ft. long is discharging water at a velocity of 6 ft. per sec. under a head of 40 ft. At the lower end of the conduit is a surge chamber 36.6 ft. in diameter. If a valve at the lower end of the conduit is suddenly closed, how high will water rise in the surge chamber and how long will it take to reach this maximum? Consider the conduit rigid and the water incompressible. *Ans.* (*a*) 18.0 ft.; (*b*) 460 sec.

6. If the rise of pressure in a 30-in. steel water pipe is 15 lb. per sq. in. owing to sudden closure of a valve, what was the rate of flow in the line before closure? Assume wall thickness of pipe to be $\frac{1}{2}$ in. and water temperature 68 deg. Fahr. *Ans.* 1.54 cu. ft. per sec.

7. A 2-in. galvanized-iron pipe is supplied with water at 60 deg. Fahr. under a head of 100 ft. The pipeline is 2000 ft. long and is terminated by a valve. Assuming entrance and exit loss coefficients 0.4 and 0.6, respectively, compute the maximum pressure at the end of the line if a valve is closed suddenly. Use Reynolds curve and indicate the steps. At what time after closure will the pressure first reach a maximum at a point 500 ft. before the valve? *Ans.* (*a*) 194.3 lb. per sq. in.; (*b*) 0.118 sec.

8. Two water reservoirs are connected by 2000 ft. of 8-in. pipe followed by 1000 ft. of 6-in. pipe. The pipe is clean cast iron and the difference in surface elevation is 100 ft. (*a*) Compute the steady rate of discharge between the reservoirs. (*b*) If a valve is suddenly closed at the lower end of the 6-in. pipe, what will be the instantaneous pressure rise if the velocity of a pressure wave is 3800 ft. per sec.? (*c*) If the valve is then suddenly

opened, how long will be required for the velocity to reach 90 per cent of the velocity during steady flow?

9. A pipeline consists of 4-in. standard wrought-iron pipe. Compute the velocity of a pressure wave in the conduit when carrying water. The line is 1500 ft. long and discharges under a static head of 200 ft. Compute the pressure rise for full closure in a time less than the first half cycle.

10. The quantity of water flowing in a pipeline is to be determined by measuring the pressure rise for sudden closure of a valve at the end of the line. The pipe is wrought steel, inside diameter 30 in., wall thickness $\frac{1}{2}$ in. If the pressure rise is 1000 lb. per sq. ft., what was the initial discharge in cubic feet per second? Assume water temperature of 68 deg. Fahr.

11. The outlet valve from a water main 5000 ft. long is closed in 1 sec. Determine the rise in pressure if the initial velocity of flow is 4 ft. per sec., and the pipe is rigid. *Ans.* 254.8 lb. per sq. in.

12. A water-hammer wave is generated by the sudden closure of a valve in a 3-in. standard wrought-iron pipe. The initial velocity of flow was 4 ft. per second. At a point 200 ft. above the valve, the pipe changes to 4-in. standard wrought-iron pipe. Compute (*a*) velocity of pressure wave in 3-in. and 4-in. pipes; (*b*) time interval for wave to travel from valve to change in diameter; (*c*) pressures above and below the change in diameter just after passage of the wave.

13. A pipeline is 3000 ft. long and 12 in. in diameter. The velocity of steady flow is 5 ft. per second. If the pipe is rigid, compute the maximum pressure attained if the outlet valve is closed at a uniform rate in 5 sec.

14. A very large air receiver under 100 lb. per sq. in. pressure is suddenly connected to a small pipeline in which the air is initially stationary and under atmospheric pressure. Compute the time for the pressure wave to travel 500 ft. and the fluid velocity induced by the pressure wave.

15. An expanding nozzle of the proportions shown in Fig. 127 has a throat diameter of 0.25 in. The air pressure in the upstream chamber is 100 lb. per sq. in. and the temperature 150 deg. Fahr. Compute the maximum weight rate of discharge and the corresponding downstream pressure for complete expansion. Draw the pressure line for full expansion. Determine whether compression shocks will occur for downstream pressures of 30, 50, and 70 lb. per sq. in. and draw the pressure curves for these conditions. Compute the loss of mechanical energy and the change in entropy for each condition.

CHAPTER IX

FLOW IN OPEN CHANNELS

The distinguishing feature of the flow of water in open channels, as contrasted with pipe flow, is that the cross-sectional area is free to change in accordance with the dynamic conditions instead of being fixed. The subject is a very broad one and only its elementary quantitative relationships will be covered in this chapter. The treatment applies to any liquid of small viscosity.

STEADY FLOW

114. Uniform Flow in Channels of Small Slope.—The term *steady flow* has been defined as a condition in which the velocity

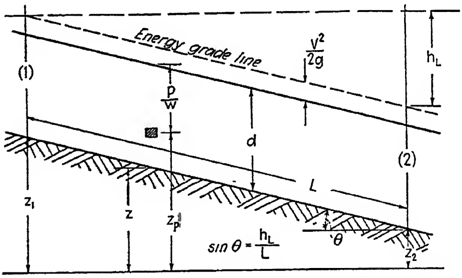


FIG. 128.—Open channel with small slope (slope exaggerated).

at any point does not vary with time. *Uniform flow* refers to flow with unchanging cross section. Steady, uniform flow is a type commonly assumed in computations, although flow of this type is probably rare in nature. Referring to Fig. 128, the hydraulic grade line for this type of flow coincides with the water surface, and since the velocity is constant, the hydraulic and energy grade lines are parallel to the bottom of the channel. Strictly speaking, the slope S should be the sine of the angle of inclination rather than the tangent but the analysis applies only to small slopes for which the sine and tangent are very nearly equal.

At any point in the channel, the summation of pressure head and elevation is

$$\frac{p}{w} + z_p = z + d$$

Bernoulli's equation written between any two points is

$$z_1 + d_1 + \frac{V_1^2}{2g} = z_2 + d_2 + \frac{V_2^2}{2g} + h_L$$

From the conditions for uniform flow,

$$d_1 = d_2, \qquad V_1 = V_2, \qquad z_1 - z_2 = h_L$$

This is merely a restatement of the parallelism of the bottom and the energy grade line. Flow in long channels is approximately uniform except for short lengths near the ends, and the computations of such flows are largely concerned with estimating the friction loss for a given velocity or discharge, or the discharge for a given head loss.

It is to be noted particularly that the hydraulic and energy grade lines are parallel to the bottom only during steady uniform flow. Incorrect usage of the equations for friction loss has led to many erroneous conclusions regarding unsteady and non-uniform flow. *In general, the friction equations give the slope of the energy grade line and it is only under the special conditions stated above that this is the same as the slope of the bottom and of the hydraulic grade line.*

115. Hydraulic Radius.—Because of the number of different forms used, the development of a single characteristic specifying the size of an open channel is of great importance. In connection with flow in pipes, the *hydraulic radius* was mentioned as a means of applying data on circular pipes to pipes of irregular section. This quantity was first used in friction computations for open channels and it is here that it finds its greatest usefulness. The quantitative definition of the hydraulic radius is

$$\text{Hydraulic radius} = \frac{\text{cross-sectional area}}{\text{wetted perimeter}}$$

The area is that below the water surface and the wetted perimeter is the peripheral length of solid boundary in contact with the water. The hydraulic radius derives its significance from the

fact that the component of the weight tending to maintain motion is proportional to the area while the resisting force is approximately proportional to the wetted perimeter. The ratio of area to wetted perimeter is therefore a linear dimension specifying the hydraulic size as regards friction only. On the basis of the data and theoretical analysis presented in Chap. IV, it appears that the hydraulic radius may be expected to represent the size as regards friction in conduits and open channels, if the main mass of liquid moves with relatively uniform velocity, and the hydraulic roughness is uniform around the periphery. In artificial channels in which the depth and width are approximately of the same magnitude, the hydraulic radius satisfactorily represents the hydraulic size, but in wide channels having irregular bottoms, such as rivers in the "overbank" condition, it does not appear to have much significance when applied to the channel cross section as a whole.

116. Critical Velocity for Laminar Flow.—The range of depth, velocity, and viscosity in the flow of water in open channels is in general such that the flow is turbulent. In laboratory work and in certain specialized field problems, such as the "sheet flow" which is of importance in soil erosion, the flow may be laminar. Allen's investigations showed that the flow was always turbulent with $VR/\nu > 4000$ and that the critical Reynolds number decreased with increasing hydraulic radius and roughness. The lower critical value at which initially turbulent flow changes to laminar is of the order of 1000.

A number of observers have concluded that the smooth, glassy surface of slow-moving rivers indicates laminar flow but an examination of their data suggests another explanation. Thrupp observed this phenomenon under the following conditions:

River	Depth, ft.	Average velocity, ft./sec.	Approximate Reynolds number, VR/ν
Thames	7.6	0.665	480,000
Kennet	2.4	0.64	146,000

The flow was believed to be laminar because the water had a "clear, glassy nondistortive reflecting surface." Another explanation is that no capillary waves were caused by wind or

obstructions because the velocity at the surface was less than the minimum velocity of a capillary wave, which is about 0.76 ft. per. sec. Experiments at the University of California showed that below this minimum velocity, the surface near a model bridge pier or other obstruction is devoid of standing waves and presents a glassy appearance.

117. Friction Losses.—The usual equation for computing friction losses in circular pipes is

$$h_L = \frac{fL}{D}\frac{V^2}{2g}$$

The hydraulic radius of a circular pipe is equal to $D/4$. Replacing D, substituting $S = h_L/L$, and solving for V give

$$V = \sqrt{\frac{8g}{f}}\sqrt{RS}$$

This is the equation of Chezy and is normally written as

$$V = C\sqrt{RS}$$

Another method of obtaining the Chezy equation may be helpful in explaining the significance of the hydraulic radius. If it is assumed that the resistance per unit area of surface in contact with the flow may be expressed as KwV^2, the total resistance in length L is $KPLwV^2$ where P is the wetted perimeter. The component of the weight of fluid is $wALS$. For steady, uniform flow the equation is

$$wLAS = KwLPV^2$$

$$V = \sqrt{\frac{1}{K}}\sqrt{\frac{A}{P}S} = C\sqrt{RS}$$

The assumption that the tractive force at every point is related to the mean velocity in the manner specified, evidently assumes that the velocity gradient near the boundary and the roughness are everywhere the same.

In Chap. IV it was shown that the friction coefficient, f, is a function of the Reynolds number and the relative roughness. Since $C = \sqrt{8g/f}$, it follows that C must be a function of the same quantities, or in other words, C will vary with the mean velocity, hydraulic radius, roughness, and viscosity. Since

both experiments and applications are generally concerned with water at usual atmospheric temperatures, the variation in viscosity has been small and the viscosity has not been included in the equations. Another important consideration is that the friction formulas have been based largely on experiments at high values of the Reynolds number in fairly rough conduits. Experiments in pipes have shown that the coefficient tends to become constant above a value of the Reynolds number which decreases with increasing roughness. This condition of a constant coefficient was referred to as *fully developed turbulence.* It appears that most of the equations for friction losses in open channels are based on data obtained with fully developed turbulence and their applicability at low Reynolds numbers, *i.e.*, at small depths and velocities, is questionable.

1. *Ganguillet-Kutter Formula.*—Largely on the basis of the data of Humphrey and Abbot for the Mississippi River, and of Bazin for small open channels, Ganguillet and Kutter developed a formula for the coefficient C in the Chezy equation which is generally referred to as the *Kutter equation.* Its form in English units is

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \dfrac{n}{\sqrt{R}}\left(41.65 + \dfrac{0.00281}{S}\right)}$$

Here, n is a friction coefficient which is supposed to depend only on the absolute roughness of the surface of the channel. Judged simply on the basis of dimensions, this equation is nothing short of fantastic. Furthermore, Lindquist and others have thrown considerable doubt on the applicability or reliability of the original data. In spite of these more or less theoretical objections, however, the equation has proved to be reliable and has been widely used in the United States. The complex form gives little difficulty in application because graphs and tables have been prepared. The coefficient n has been found to be a reasonably accurate characterization of the hydraulic roughness of channel surfaces. Table 2 gives values of n for average conditions and is intended only to indicate the variation in this coefficient.

TABLE 2.—AVERAGE VALUES OF n FOR USE IN THE GANGUILLET-KUTTER AND MANNING FORMULAS[1]

Surface	n
Cast-iron pipe, fair condition	0.014
Galvanized wrought-iron pipe, fair condition	0.015
Smooth brass	0.011
Riveted steel pipe	0.017
Vitrified sewer pipe	0.015
Concrete pipe and flumes, average construction	0.015
Planed plank flume	0.013
Semicircular metal flumes, smooth	0.013
Semicircular metal flumes, corrugated	0.0275
Canals and ditches:	
Earth, straight and uniform	0.0225
Winding sluggish channels	0.0275
Dredged earth channels	0.030
Natural stream channels:	
Clean, straight bank, full stage	0 030
Winding, some pools, and shoals	0 040
Same, but with stony sections	0 055
Sluggish reaches, very deep pools, very weedy	0.070–0.125

[1] From "Handbook of Hydraulics" by H. W. King.

For values to use in design, a report by Fred C. Scobey is recommended. A very thorough comparison of friction formulas was made by Houk, who concluded that the Kutter formula best represented field measurements.

2. *Manning Equation.*—The equation developed by Robert Manning is simpler in form than the Kutter equation and lends itself admirably to analytical treatment of flow problems. This equation has the form

$$C = \frac{1.486}{n}R^{\frac{1}{6}}$$

Here, C is the coefficient in the Chezy equation and n has the same value as in the Kutter equation. A comparison of the Kutter and Manning equations by Scobey showed that the experimental values of n for the two equations were very nearly the same throughout the range of values usually found in flumes.

3. *Bazin Equation.*—The formula proposed by Bazin for the coefficient C in the Chezy equation is, in English units,

$$C = \frac{157.6}{1 + \quad m}$$

In this equation R is the hydraulic radius and m is a coefficient representing the absolute roughness.

A great many other formulas for friction losses have been proposed, many of them being of the exponential type. On the basis of recent advances in the theory of turbulence and hydraulic roughness, Lindquist has developed formulas having a good theoretical background which agree well with field experiments. Since all of these equations include empirical coefficients, the best equation to use is that which is backed by the greatest range of experimental data and in this respect the Kutter equation is decidedly advantageous. For general analysis, such as is presented in this chapter, the Manning equation is preferable.

Example: A canal of trapezoidal section has a bottom width of 10 ft., and side slopes making an angle of 60° with the horizontal. The bottom slope is 0.002 and the depth of flow is 7 ft. The canal is lined with smooth concrete. Compute the discharge by the Manning equation assuming steady, uniform flow.

Top width of channel:

$$10 + 2 \times 7 \cot 60° = 10 + 14 \times 0.577 = 18.08 \text{ ft.}$$

Area of channel:

$$\frac{10 + 18.08}{2} \times 7 = 98.3 \text{ sq. ft.}$$

Wetted perimeter:

$$10 + 2 \times 7 \csc 60° = 10 + 14 \times 1.155 = 26.17 \text{ ft.}$$

$$R = \frac{98.3}{26.17} = 3.76 \text{ ft.}$$

$$Q = AV = 98.3 \frac{1.486}{0.015} 3.76^{\frac{2}{3}} 0.002^{\frac{1}{2}}$$

$$= 98.3 \frac{1.486}{0.015} \times 2.42 \times 0.0447$$

$$= 98.3 \times 11.0 = 1080 \text{ cu. ft. per sec.}$$

118. Velocity Distribution in Open Channels.—It is an experimental fact that the maximum velocity in a straight reach of open channel may occur at a point below the water surface and frequently it does not occur at the center. No quantitative

explanation of this phenomenon has yet been advanced and it is in direct conflict with the previously mentioned theories of tractive force and turbulent mixing. Surface tension or wind effects can not explain the relatively large apparent force retarding the laminae above the maximum velocity. In one respect, the free surface corresponds to a solid boundary, namely, that the *mixing length* of the turbulence can not exceed the distance to the surface, but on the other hand, the free surface can not exert a tractive force. The assumption of secondary spiral flows with the flow outward at the bottom and inward at the surface on each side of the center line gives results which are roughly in agreement with the observed depression of the maximum velocity. The longitudinal length of this spiral motion is of the order of 50 times the width of the channel. However, the existence of such spiral motion has not been definitely established experimentally and the theory has not been worked out in detail so that the subject can be treated at present only from the empirical viewpoint.

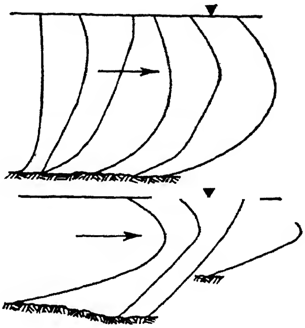


Fig. 129.—Typical vertical velocity distribution curves in open

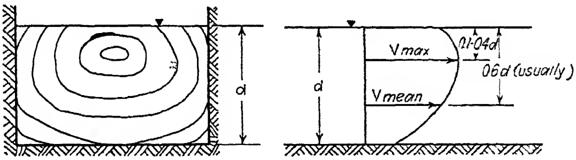


Fig. 130.—Typical velocity distribution in long straight channel.

Figure 129 shows typical velocity curves as reported by Scobey. Figure 130 represents an average velocity distribution in a long straight channel in which the effect of obstructions and disturbances has been eliminated by friction. Figure 131 gives average conditions in straight channels having different ratios of width to depth as obtained by Lane. There appears to be a tendency towards depression of the thread of maximum velocity as the width-depth ratio decreases and, in fact, in some laboratory

channels at a depth several times the width, the maximum velocity has been observed at or below mid-depth.

At bends in channels, the high-velocity filaments tend to move to the outer bank, forcing the slow-moving filaments to the

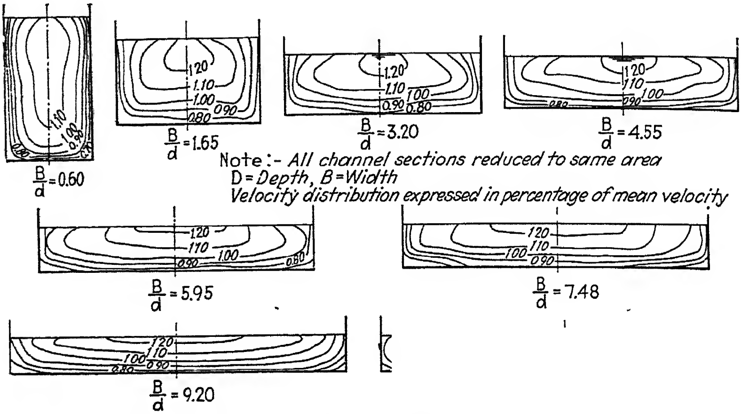


FIG. 131.—Effect of ratio of width to depth on velocity distribution in rectangular channels. (*E. W. Lane, "Stable Channels in Erodible Material," Proc. Am. Soc. C. E., November*, 1935, *p.* 132.)

inside. If the maximum velocity occurs some distance below the surface, the dynamic pressure at the outside may result in an upward movement with water "boiling" up to the surface and

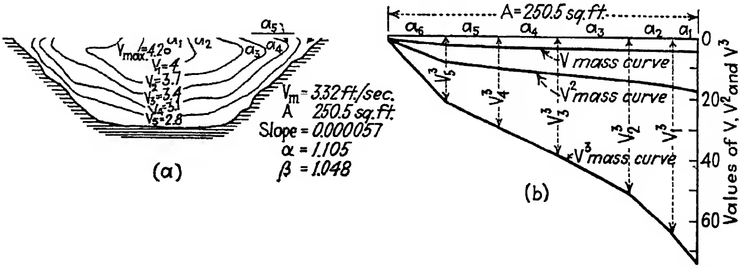


FIG. 132.—(*a*) Typical velocity distribution in trapezoidal channel. (*b*) Mass curves used in determining the value of α for the velocity distribution shown in (*a*).

then spreading out towards the middle of the channel. In wide, shallow channels the surface currents generally move toward the outer bank and the bottom currents towards the inner bank, causing the formation of sand bars at the inside of the bend.

119. Velocity Head, Momentum.—In Chap. VII, it was found that the true velocity head to be used in Bernoulli's equation is the average kinetic energy per unit weight of water flowing. Consequently, $V_m^2/2g$, where V_m equals Q/A, is less than the true value by an amount which depends upon the velocity distribution. Following the same line of reasoning, the corrective factor α is obtained by graphical solution of the equation

$$\alpha = \frac{\int V^3\,dA}{V_m^3 A}$$

where V is the measured velocity over a differential area dA. Figure 132 shows a typical velocity distribution and the mass diagram used to obtain the value of α graphically. The value of α is ordinarily around 1.05 to 1.10 but may attain much higher values if the velocity distribution is greatly distorted, as in Fig. 133. Rehbock has shown that α is given approximately by $\alpha = 1 + \epsilon^2$, where the maximum velocity equals $(1 + \epsilon)V_m$.

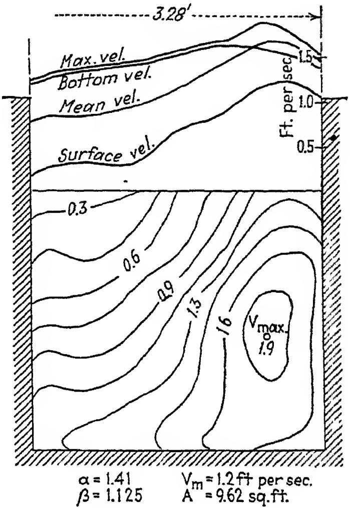


FIG. 133.—Large distortion of velocity distribution in an open channel, producing unusually high values of α and β.

In solving flow problems by means of the momentum equations, the momentum per second carried past a certain section must be computed. If the velocity distribution is not uniform, the equation for the momentum per second is

$$M = \frac{w}{g}\int_0^A V^2\,dA = \beta\frac{w}{g}AV_m^2$$

Here, β is a corrective coefficient depending for its value on the velocity distribution. Approximately,

$$\beta = 1 + \frac{\epsilon^2}{3}$$

Values of α and β for a number of different conditions appear in Table 3.

TABLE 3.—VALUES OF ENERGY AND MOMENTUM COEFFICIENTS

Channel dimensions				Hydraulic elements		Coefficients			Remarks
						α		β	
Width, ft.	Maximum depth, ft.	Hydraulic radius, ft.	Area, sq. ft.	Critical depth, ft.	Mean velocity, ft./sec.	Graphical	Rehbock	Graphical	
1.97	2.83	0.73	5.59	0.65	1.05	1.20	1.10	1.07	Rectangular channel 3 ft. above weir and obstructions upstream. (Fig. 133)
3.28	2.83	1.067	9.64	0.71	1.16	1.22	1.20	1.08	
3.28	2.87	1.066	9.62	0.72	1.20	1.41	1.37	1.12	
3.3	1.41	0.76	4.65	1.63*	8.41	1.07	1.04	1.03	Simplon Tunnel. Center of straight reach 164 ft. long
34.6	10.6	6.11	250.5	4.67	3.32	1.10	1.07	1.05	Fig. 132
6.52	4.92	2.07	31.2	2.52	4.88	1.07	1.03	1.034	Horseshoe conduit. Straight reach
523.	12.51	8.0	4365.	6.27	3.36	1.35	1.43	1.121	Rhine 1200 ft. below bridge. Long curve
8.50	4.54	2.28	36.92	2.25	2.91	1.06	1.02	1.01	Sudbury Aqueduct†
8.75	4.01	2.13	32.4	2.16	2.87	1.04	1.04	1.014	Sudbury Aqueduct†
9.00	3.00	1.8	23.59	1.97	2.60	1.04	1.03	1.014	Sudbury Aqueduct†
8.90	2.03	1.35	15.24	1.74	2.16	1.04	1.02	1.01	Sudbury Aqueduct†
8.75	1.51	1.07	10.92	1.64*	1.87	1.04	1.03	1.012	Sudbury Aqueduct†
......	0.869	.....	0.575	1.17*	7.59	1.161	....	.....	Computed from Bazin's Series 10
......	0.803	.....	0.395	0.468	0.675	1.138	....	.....	Computed from Nikuradse's data
4.22	2.5	.....	....	.....	.....	1.07	....	.....	Series (*E*). Schoder and Turner
4.22	5.0	.....	....	.....	.....	1.08	....	.....	Series (*I*). Runs 54 to 58
4.22	5.0	.....	....	.....	.....	1.6	....	.....	Series (*I*)
4.22	5.0	.....	....	.....	.....	2.08	....	.....	Series (*I*)
4.22	10.1	.....	....	.....	.....	1.8	....	.....	Series (*D*). Runs 101 to 105
4.22	9.0	.....	....	.....	.....	2.0	....	.....	Series (*D*), (*L*), (*M*).

* Maximum depth less than critical depth.
† Bottom slope = 0.000189.

In all the equations for friction losses, the slope term is the slope of the energy grade line and the value of α assumes considerable importance in the determination of the friction coefficients in short lengths of channel. The general expression for the slope is

$$S = \frac{\Delta(z + d) + \dfrac{\alpha_1 V_{m1}^2 - \alpha_2 V_{m2}^2}{2g}}{L} \tag{9.1}$$

Here, $\Delta(z + d)$ is the change in surface elevation in length L and the subscripts 1 and 2 refer to the two ends of the reach. Equation (9.1) points to the desirability of obtaining the velocity distribution at both ends of an experimental section unless the flow is uniform or the reach is very long.

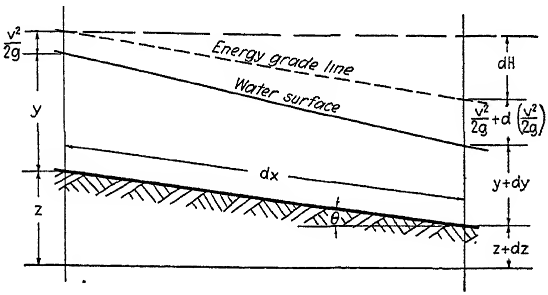


FIG. 134.

Other applications of the velocity-head correction are as numerous as the problems that can be solved by the Bernoulli equation. The magnitude of the correction ranges from 4 to 40 per cent under conditions that are frequently met in practice, and may exceed 100 per cent, as in the experiments of Schoder and Turner (Table 3).

120. Nonuniform Flow in Channels of Constant Shape.—It will be assumed that the channel is prismatic, *i.e.*, the shape does not change along its axis, that the slope is constant or varies so gradually that centrifugal forces are negligible, and that the slope is small enough that the pressure head does not differ appreciably from the depth. Bernoulli's equation for any section is then

$$H = z + y + \frac{V^2}{2g}$$

Differentiating,

$$dH = dz + dy + \alpha\frac{V\,dV}{g}$$

The friction loss according to the Chezy equation is

$$dH = -\frac{V^2}{C^2R}dx$$

Inserting this value, collecting terms, and dividing through by dx,

$$\frac{z}{dx} + \frac{dy}{dx} =$$

Following the usual convention regarding the sign of the bottom slope, $dz/dx = \sin\theta = -S_0$. Also $V = Q/A$ and

$$\frac{}{g}\,\frac{}{dx} \qquad \frac{}{dx} \qquad -\frac{Q^2}{gA^3}\,\frac{ay}{dx}$$

Here, b equals the surface width and is therefore equal to dA/dy. Substituting in Eq. (9.2) and solving for the rate of change of depth,

$$\frac{dy}{dx} = \frac{S_0 - \dfrac{Q^2}{C^2A^2R}}{1 - \alpha\dfrac{bQ^2}{gA^3}} \tag{9.3}$$

Equation (9.3) is the differential equation for flow with varying depth in a prismatic channel.

121. Integration of the Nonuniform Flow Equation.—Analytical methods of integrating the nonuniform flow equation for special forms have been developed by Tolkmitt, Bresse, Ruehlmann, and others. Bakhmeteff has prepared tables which are intended to be applicable to any channel form. All of these methods suffer in generality from the fact that in one way or another an average value of C or its equivalent is assumed and none of them take into consideration the coefficient α. As an example of the methods used in integrating the differential equation to obtain a "backwater function," Eq. (9.3) will be applied to a wide, rectangular channel for a range of depths in which the Chezy coefficient, C, is approximately constant. If y_0 is the depth at which uniform flow would occur with the same

discharge Q and bottom slope S_0, the value of y_0 given by the Chezy equation is

$$y_0 = \sqrt[3]{\frac{Q^2}{S_0 b^2 C^2}}$$

since $R = y_0$ for a wide channel. Substituting for Q^2 in terms of y_0 in Eq. (9.3) leads to

$$S_0\,dx = \frac{\left(\frac{y}{y_0}\right)^3 - \alpha\frac{C^2S_0}{g}}{\left(\frac{y}{y_0}\right)^3 - 1}dy$$

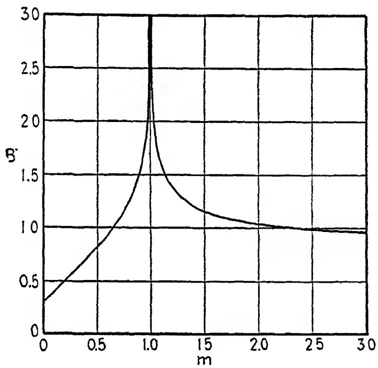


FIG. 135.—Backwater function B as a function of the depth ratio m.

Substituting

$$m = \frac{y}{y_0}, \qquad dy = y_0\,dm,$$

$$\frac{S_0}{y_0}dx = \frac{m^3 - \alpha\frac{C^2S_0}{g}}{m^3 - 1}dm = \left(1 + \frac{1 - \alpha\frac{C^2S_0}{g}}{m^3 - 1}\right)dm$$

Integrating between the limits $x = 0$ at $m = m_1$, and $x = l$ at $m = m_2$,

$$l = \frac{y_0}{S_0}\left[m_2 - m_1 + \left(1 - \alpha\frac{C^2S_0}{g}\right)\int_{m_1}^{m_2}\frac{dm}{m^3 - 1}\right]$$

Here, l is the distance between two points at which the depths

are y_1 and y_2 in a channel in which the depth of uniform flow (normal depth) is y_0. The value of the definite integral is

$$\int_{m_1}^{m_2} \frac{dm}{m^3 - 1} = \left(\frac{1}{6} \log \frac{(m-1)^2}{1 + m + m^2} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2m+1}{\sqrt{3}}\right)_{m_1}^{m_2}$$
$$= B_1 - B_2$$

Here

$$B = \left(\frac{1}{6} \log \frac{1 + m + m^2}{(m-1)^2} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2m+1}{\sqrt{3}}\right)$$

The values of B as a function of m appear in Fig. 135. The complete equation for the distance between any two depths y_1 and y_2 is

$$l = \frac{y_0}{S_0}\left[m_2 - m_1 + \left(1 - \alpha\frac{C^2 S_0}{g}\right)(B_1 - B_2)\right]$$

Example: A broad-crested weir having a discharge coefficient of 2.64 and a height of 5 ft. is built in a wide rectangular channel having a bottom slope of 0.001. The discharge per foot of width is 35.4 cu. ft. per sec. Trace the surface curve, assuming $\alpha = 1$.

$$y_0 = \sqrt[3]{\frac{q^2}{SC^2}} = 5 \text{ ft.}$$

$$\text{Head on weir} = \left(\frac{35.4}{2.64}\right)^{\frac{2}{3}} = 5.65 \text{ ft.}$$

The depth just above the weir is 10.65 ft. and the water surface slopes upward in such a manner that the depth approaches $y_0 = 5$ ft. upstream. By assuming depths between 10.65 and 5 ft., the corresponding distances can be computed.

$$y_2 = 10.65 \text{ ft.}, \qquad m_2 = \frac{y_2}{y_0} = 2.13, \qquad B_2 = 1.02$$
$$l = 5000[(2.13 - m_1) - 0.69(1.02 - B_1)]$$
$$= 5000[1.43 - m_1 + 0.69B_1]$$

y, ft.	m_1	B_1	l, ft.
10.65	2.13	1.02	0
7.65	1.53	1.14	−3450
5.65	1.13	1.51	−7220
5.00	1.00	∞	∞

In this example, tracing of the surface curve started from a point at which the depth was independently determined and the surface curve approached the normal depth asymptotically.

122. Graphical Integration.—All backwater problems for steady flow may be reduced to solving the differential equation for known values of Q and S_0. The other variables, b, A, and C are functions of y, and α is either a constant or varies with x. Equation (9.3) therefore lends itself to graphical solution if α is assumed to be constant. Rearranging the equation,

$$dx = \frac{1 - \alpha \dfrac{bQ^2}{gA^3}}{S_0 - \dfrac{Q^2}{C^2RA^2}} dy = f(y)dy$$

$$x_2 - x_1 = \int_{y_1}^{y_2} f(y)dy$$

Solution by this method consists in assuming values of y_1, computing $f(y)$, and planimetering the area under the curve between two values of y as in Fig. 136 in which

$$= \text{area } ABCD$$

The integration starts from a known depth y_1 and obtains the distance to another y_2, provided, of course, that no discontinuities in $f(y)$ occur between y_1 and y_2.

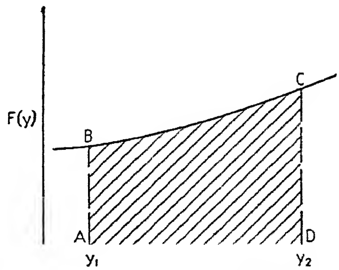


Fig. 136.—Graphical integration the backwater function.

The graphical method may be applied to any shape of channel for which the hydraulic radius is a valid indication of hydraulic size, and the proper value of C may be used for each depth. It also has the advantage of being a straightforward method which may be applied mechanically by a person unfamiliar with the theoretical background. It applies without modification to channels which slope upward in the direction of flow, in which event S_0 is negative.

123. Finite Increments.—The differential equation for non-uniform flow may also be solved by the method of finite increments as

$$\Delta x = \frac{1 - \alpha\dfrac{bQ^2}{gA^3}}{C^2RA^2}\cdot\Delta y$$

This method is more laborious than the graphical method but is useful when the channel form is not constant.

124. Normal Depth, Critical Depth, and Critical Velocity.—Certain peculiarities in the equation of nonuniform flow are of interest.

$$\frac{dy}{dx} = \frac{S_0 - \dfrac{Q^2}{C^2RA^2}}{1 - \alpha\dfrac{bQ^2}{gA^3}} \tag{9.3}$$

Assuming first that the denominator is not zero, $dy/dx = 0$ when

$$S_0 = \frac{Q^2}{C^2RA^2}$$

Representing the particular channel area which satisfies this equation by A_0,

$$= f(y_0)$$

The depth y_0 at which the slope of the energy grade line equals the slope of the bottom is called the *normal depth* because in all cases the surface curve tends to approach this depth in its extension either upstream or downstream.

If $y \neq y_0$, the theoretical rate of change of depth with distance, dy/dx becomes infinite when

$$\alpha\frac{bQ^2}{gA^3} = 1, \qquad \frac{A^3}{b} = \frac{\alpha Q^2}{g} \tag{9.4}$$

Representing the particular area which satisfies Eq. (9.4) by A_c, the corresponding depth y_c is called the critical depth. In a rectangular channel,

$$A_c = by_c, \qquad \text{and with} \qquad \alpha = 1, \qquad y_c = \sqrt[3]{\frac{Q^2}{gb^2}}$$

At the critical depth, the relationship between velocity and depth is of interest. From Eq. (9.4),

$$\frac{A}{b} = y_m = \alpha\frac{Q^2}{gA^2} = \alpha\frac{V^2}{g}$$

$$V_c = \sqrt{\frac{gy_m}{\alpha}}$$

Here, y_m is the average depth obtained by dividing the area by the *surface width*. In rectangular channels with $\alpha = 1$,

$$V_c = \sqrt{gy_m}$$

It will be shown later that this is the velocity, relative to the water, of a wave of small amplitude, and hence if the velocity

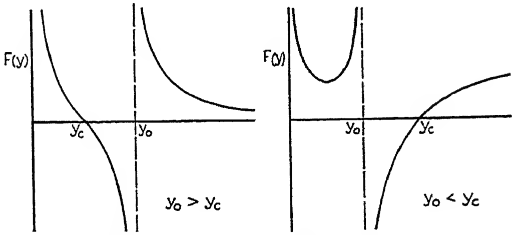


FIG. 137.—Trend of the backwater function.

increases through the critical, small waves cannot pass upstream. Therefore such a section acts as a limit to small backwater effects and it is for this reason that critical depth sections act as controls. However, waves of finite amplitude travel with a velocity greater than $\sqrt{gy_m}$ and, if sufficiently large, will pass through and drown a control. Actually, of course, at the critical depth the surface does not rise vertically ($dy/dx = \infty$) because the vertical acceleration would be appreciable and the analysis is no longer valid since the pressure head at the bottom would not equal the depth, which was one of the initial assumptions.

When the numerator and denominator of Eq. (9.3) are both zero simultaneously, the normal depth and the critical depth coincide and dy/dx may be finite.

Figure 137 shows the variations in the function $F(y)$ in relation to the critical and normal depths for a certain rate of discharge in a channel of known shape.

125. Control Points.—In tracing backwater curves it is necessary to establish the elevation of one point by some independent

method and for convenience such a point will be referred to as a *control.*

The simplest type of control is a level of fixed elevation such as the surface of an ocean, lake, or large reservoir. Here, the elevation may even change periodically or progressively without modifying the nature of the control, provided only that the elevation is independent of the instantaneous rate of discharge in the tributary channel under investigation. The situation may become very complex as in tidal rivers but the essential elements remain the same in that the tide stage at the mouth is independent of the river discharge.

The second type of control is provided by all those situations and arrangements of structures which cause a unique relationship to exist between the water-surface elevation and the rate of discharge. Weirs of all types, gates, sluices, narrows, rapids, free overfalls, are all examples of potential control points but none of them may be regarded as absolute controls under all conditions. A weir which acts as a control at one discharge may be drowned out by the backwater from a control downstream at higher flows. Certain sections, such as high spillways and falls, are evidently controls at all discharges, but if there is doubt, one must project the backwater curve upstream from a lower control.

Loosely speaking, a critical depth might be said to occur at any point at which the depth is determined uniquely by the discharge. However, it appears to be preferable to confine usage of the term *critical depth* to the conditions indicated in the derivation of Eq. (9.4). It might also be mentioned that Eq. (9.4) gives an average critical depth for a whole cross section and has significance only when the velocity over the cross section is reasonably uniform. For example, if a channel consists of a deep, narrow portion and a broad, shallow one, the critical depth for the channel as a whole has little significance.

Bearing in mind these restrictions, it may be said that critical-depth control points may occur at or near changes in bottom slope, provided that

$$S_{01} < S_c < S_{02} \tag{9.5}$$

where S_{01} is the slope of the bottom above the break, S_{02} is the slope below the break, and S_c is that slope necessary to maintain

uniform flow at the critical depth. If the inequality (9.5) is satisfied, the depth upstream from the break in grade, but very near to it, may be computed from Eq. (9.4), provided that the section is not drowned out by the backwater from the next control downstream.

126. Specific Energy.—The term *specific energy* is used to indicate the elevation of the energy grade line above the channel bottom at a particular section. It may increase, decrease, or remain constant in the direction of flow depending upon the slope of the bottom and of the energy grade line. It should not be confused with the total energy.

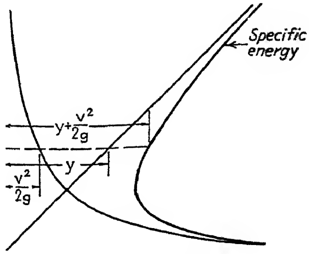
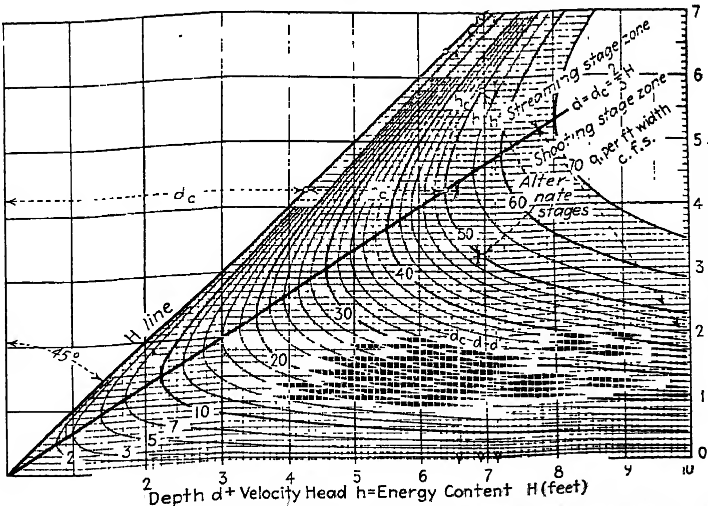


FIG. 138.—Construction of the specific energy curve.

The equation for specific energy in a channel of small slope is

$$E = y + \alpha\frac{V^2}{2g} = y + \alpha\frac{Q^2}{2gA^2} \tag{9.6}$$



-Energy curves for flow per unit width in rectangular channels. (*Fred C. Scobey, "Flow of Water in Flumes," U. S. Dept. Agr. Bull.* 393, p. 72.)

If Q is considered to be a known constant, as it is in a certain group of problems, then E is a function of y and the shape of the

channel. The method of drawing the curve is shown schematically in Fig. 138. The curves for specific energy in rectangular channels as drawn by Scobey are shown in Fig. 139. At every value of the specific energy, there are two possible depths of flow for any value of Q, except at one particular depth which corresponds to the minimum specific energy. The two possible depths are referred to as the *alternate stages*, the smaller depth (and higher velocity) being the stage of *shooting flow* and the greater, that of *streaming flow*.

The depth giving a minimum specific energy for any value of Q may be shown to be identical with the critical depth, as given by Eq. (9.4), as follows: Differentiate Eq. (9.6) with respect to y, assuming Q as constant, to obtain the minimum value of E.

$$\frac{dE}{dy} = 1 - \alpha\frac{Q^2}{gA^3}\frac{dA}{dy} = 0$$

$$\frac{dA}{dy} = b = \text{surface width}$$

Therefore, E is a minimum when $\frac{A^3}{b} = \alpha\frac{Q^2}{g}$ which is the same condition as is indicated by Eq. (9.4).

At the critical depth the specific energy is a minimum consistent with the discharge Q, or conversely, at this depth the discharge is the maximum consistent with the specific energy.

127. Momentum Curve.—Use of the momentum principle in solving fluid problems was considered in Chap. III and the hydraulic jump, an open-channel phenomenon, was treated by this method. A number of similar channel problems may be treated in this way and the method will now be amplified. It was found that one necessary condition is that the pressures either be known or act on surfaces parallel to the direction of the summation of forces. On the basis of this requirement the momentum equations should be applied only to prismatic channels of small slope. If the channel changes shape, there is an unknown component of the pressure at the sides acting in the direction of motion. If the channel has an appreciable slope, one must know either the weight of fluid between the sections or the pressure on the bottom, and in general, neither of these quantities can be obtained analytically.

As applied to steady flow, the momentum equation is

$$P_1 + \beta_1 \frac{Qw}{g} V_1 = P_2 + \beta_2 \frac{Qw}{g} V_2 \tag{9.7}$$

The distance between points (1) and (2) is usually small and the friction forces can generally be neglected, as in Eq. (9.7). The quantity β is the momentum coefficient discussed in Sec. 119. If point (1) is upstream, β_2 is generally greater than β_1 and both β's exceed unity. The pressure force P is computed from the relationship

$$P = w \int_0^y bz\, dz$$

Here, y is the maximum depth and b is the width at distance z below the surface. The integral gives the statical moment of

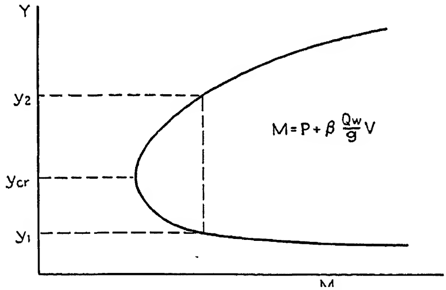


FIG. 140.—The momentum curve.

the cross section about the water line and therefore in a channel of any shape,

$$P = wz_0A$$

where z_0 is the distance down to the center of gravity of the area. In a rectangular channel,

$$P =$$

If Q is considered as a known constant, the summation of pressure force and momentum per second may be plotted as a function of y. Making

$$M = P + \frac{\beta Qw}{g} V$$

the curve constructed by assuming values of y and computing M has the form shown in Fig. 140. Solution of the hydraulic jump

consists in satisfying the relationship $M_1 = M_2$, as for the depths y_1 and y_2 in the diagram. The curve has a minimum value which can be shown to occur at the critical depth in the following manner:

$$M = wz_0A + \frac{\beta wQ^2}{gA}$$

$$\frac{dM}{dy} = 0 = wA\,\frac{dz_0}{dy} + wz_0\,\frac{dA}{dy} - \frac{\beta wQ^2}{gA^2}\,\frac{dA}{dy} \tag{9.8}$$

$$\frac{dA}{dy} = b$$

To evaluate dz_0/dy, assume a small increment of depth, dy, and take the statical moment about the water surface.

$$(z_0 + dy)A + \frac{b\,\overline{dy}^2}{2} = (A + b\,dy)(z_0 + dz_0)$$

$$\frac{dz_0}{dy} = 1 - \frac{bz_0}{A}$$

Substituting in Eq. (9.8) and canceling common terms, the condition for a minimum value of M is that

$$0 = 1 - \beta\frac{bQ^2}{gA^3}$$

$$\frac{A^3}{b} = f(y) = \beta\frac{Q^2}{g}$$

If $\beta = \alpha$, the minimum in the momentum equation occurs at the depth giving the minimum energy and that at which a discontinuity in the backwater curve occurred (see Sec. 124). Both α and β have generally been assumed as unity in analyzing the hydraulic jump and related phenomena.

With proper modifications the momentum principle may also be applied to problems in unsteady flow.

128. Hydraulic Jump.—The agreement between theoretical and experimental results for an hydraulic jump in a horizontal, prismatic channel is surprisingly good. This phenomenon has attracted the interest of a number of investigators and the reader should consult the writings of Hinds, Stevens, and Bakhmeteff, among others, for additional details. The subject was treated briefly in Chap. III where it was shown that a loss of

available mechanical energy is involved and that solution must be made by the momentum principle.

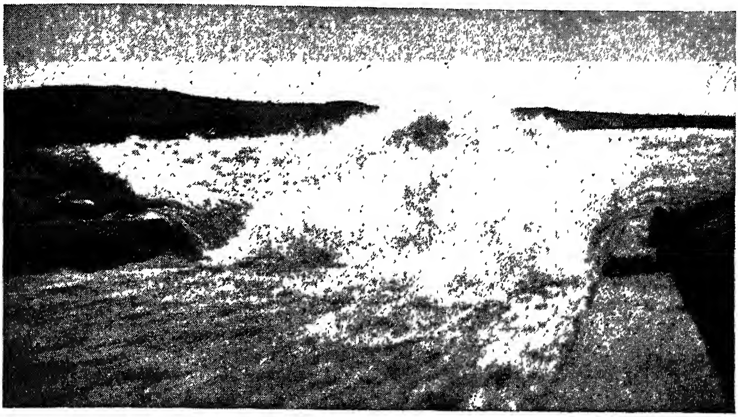


FIG. 141.—Hydraulic jump at end of chute entering Lake Newell. (*Courtesy Canadian Pacific Railway.*)

The method of predicting the depth, y_2, after the jump is to set $M_1 = M_2$ and solve the equation analytically or graphically. A slight bottom slope or gradual variation in channel form does

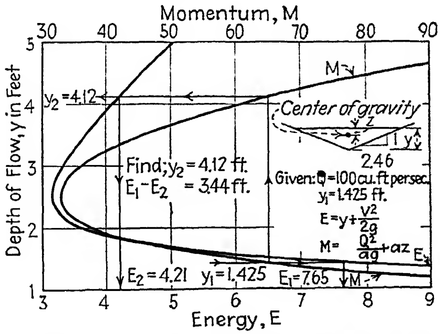


FIG. 142.—Momentum and energy curves for graphical solution of hydraulic jump in the triangular channel shown. (*G. H. Hickox, Civil Engineering, May, 1934.*)

not change the jump markedly from its dimensions in the assumed type of channel but does have a considerable influence on its location.

In Fig. 142, application of the momentum and energy equations is made to the prediction of the depth y_2 and the loss of

energy in an hydraulic jump in an open channel. The two depths are fixed by the momentum equation and the energy loss is the

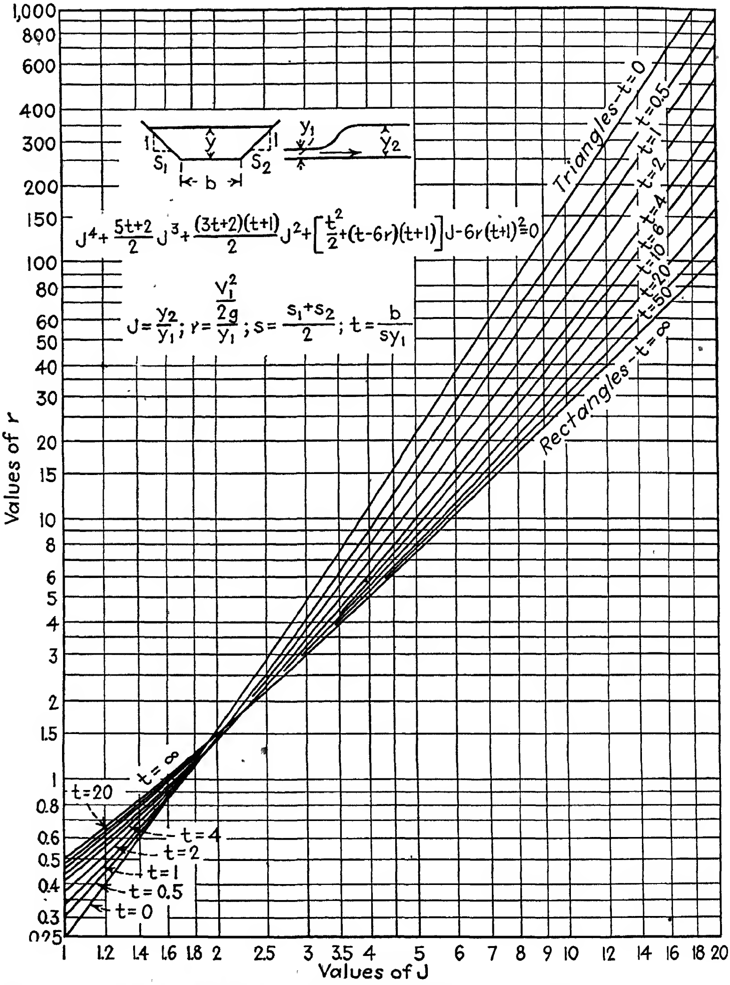


FIG. 143.—Chart for solving hydraulic jump in trapezoidal channels. (*G. H. Hickox, Civil Engineering, May,* 1934.)

difference in the specific energy of these depths. On the basis of momentum alone, the depth might decrease abruptly from y_2 to y_1 but this would require a spontaneous creation of energy and

is impossible. Figure 143 is a general diagram, based on momentum, for solution of the hydraulic jump in triangular, trapezoidal, and rectangular channels. The computed depth y_2 is that which must be fixed by the backwater curve from the next control downstream if a jump is to occur.

The horizontal dimensions of the hydraulic jump in a horizontal, rectangular channel have been studied by Bakhmeteff, who has prepared a dimensionless diagram for predicting the length of the rising surface in terms of the hydraulic elements. The rate at which the bottom velocity decreases in passing

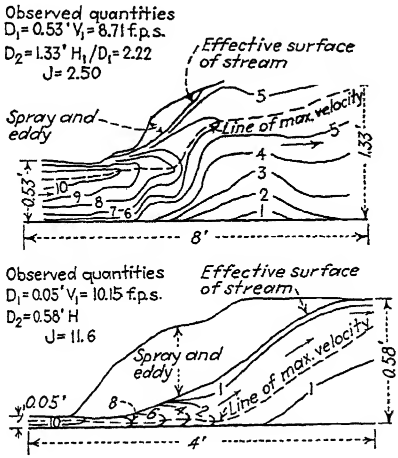


FIG. 144.—Longitudinal sections of hydraulic jumps showing lines of equal velocity. Numbers on lines are velocities in feet per second. (*Ross M. Riegel and John C. Beebe, "The Hydraulic Jump as a Means of Dissipating Energy," State of Ohio, Miami Conservancy District, Tech. Rept., Part* III.)

through the jump is a feature more important than the surface curve because of its relation to scouring of the bottom. This feature of the jump was investigated by the Miami Conservancy District. The results are shown in Fig. 144. The main stream of fluid is seen to *separate* from the bottom.

One feature of the hydraulic jump which appears to have been overlooked is that, although $M_1 = M_2$, the lines of application of the pressure forces and the inertial reactions before and after the jump do not coincide. Consequently, the water passing through the jump is subjected to a moment which depends for its magnitude on the change in depth. This moment reverses

the direction of the surface currents and causes the familiar eddy on the surface. The water passing through the jump is given a rotation, and below the jump the summation of the energy is the same as above it but a portion of the energy is in the form of kinetic energy of rotation and is not available to maintain the flow. Analysis of this feature of the jump has not been worked out in detail, but qualitatively it explains the fact that energy is "lost" in frictionless flow and the fact that the jump loses much of its effectiveness in the dissipation of energy below a spillway or chute if it is *drowned* by the backwater downstream.

FIG. 145.—Flow at a transition from a trapezoidal canal to a semicircular flume. (*Courtesy Fred C. Scobey, U. S. Bur. Agr. Eng.*)

129. Q-Curves.—In Eq. (9.6) the discharge Q was considered as a constant and the specific energy E was computed as a function of y. In a certain class of problems, it is more convenient to treat E as known and draw diagrams of discharge as a function of y. These diagrams were introduced by Koch and are generally referred to as *Q-curves*. Solving Eq. (9.6) for Q,

$$Q = A\sqrt{\frac{2g}{\alpha}(E - y)} \tag{9.9}$$

Since A is a function of y such that $A = 0$ when $y = 0$, it is evident that $Q = 0$ when $y = 0$ and $y = E$, and must therefore have a maximum value at some value of $y < E$, which from the previous analysis is known to be the critical depth.

In rectangular channels, $A = by$ and Eq. (9.9) may be written as

$$\left(\frac{y}{E}\right)^3 - \left(\frac{y}{E}\right)^2 + \frac{\alpha Q^2}{2gE^3} = 0$$

$$q = ME^{\frac{3}{2}}, \qquad M = \sqrt{\frac{2g}{\alpha}}\left[\left(\frac{y}{E}\right)^2 - \left(\frac{y}{E}\right)^3\right]^{\frac{1}{2}} \tag{9.10}$$

Here, q is the discharge per unit width (Q/b). Values of M as a function of y/E for $\alpha = 1$ are as follows:

y/E	M	y/E	M
0.00	0.000	0.65	3.086
0.05	0.396	0.667	3.089
0.10	0.761	0.70	3.077
0.20	1.436	0.80	2.871
0.30	2.014	0.90	2.280
0.40	2.486	0.95	1.705
0.50	2.837	1.00	0.000
0 60	3.045		

The quantity M, and therefore q, reaches a maximum at $y = \frac{2}{3}E$, which is the critical depth. The Q-curve for any value of E may be constructed by multiplying the tabular values by $E^{\frac{3}{2}}$ and plotting against the corresponding value of y.

130. Solution of Flow Problems.—In applying Q-curves, curves of specific energy, backwater equations, and other methods mentioned in this chapter, one must remember that they are based upon assumptions which restrict their field of application. They all apply only to small slopes. The general equation of surface slope [Eq. (9.3)] and the momentum curve apply only to channels of constant form. Except for the momentum curves they are merely rearrangements of Bernoulli's equation and their application requires all of the information necessary for the solution of that equation.

To give numerical examples of even the important situations which may be treated by one or more of these methods is beyond the scope of this text. However, one point of considerable practical importance is the proper use of the Q-curves and curves of specific energy and this point will be illustrated by an example.

Example: A broad, rectangular channel is obstructed by a weir of the form shown in Fig. 146. The upstream and downstream slopes are very gradual and as a first approximation the flow will be considered as frictionless. No backwater effect is exerted by the next control downstream.

Case I. The Elevation of the Energy Grade Line Is Known.—This assumption corresponds to flow from a large reservoir of known surface elevation. The value of E, the specific energy at any section, is $(H - z)$. The Q-curves for each section are drawn as shown by using Eq. (9.10). From continuity, the value of Q is constant and therefore at all stations there are two possible depths of flow except at the control section, which is the section showing the minimum of all the individual maximum points on

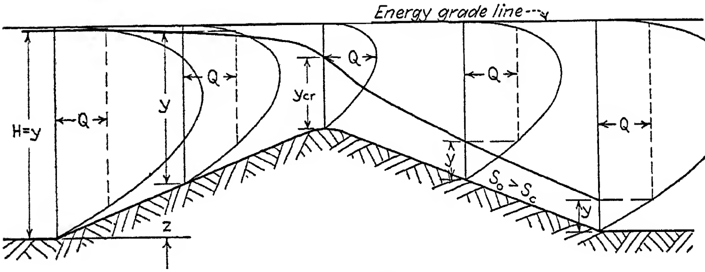


FIG. 146.

the Q-curves. Under the assumed conditions of frictionless flow and gradual slopes, the control section is at the crest. This line of reasoning has merely set a maximum value for the discharge but has not shown that a smaller discharge might not occur. Some writers treat the problem as one of least work and assume that the critical depth must occur because it gives the maximum discharge consistent with the energy available. However, such an assumption is unnecessary because application of Eq. (9.3) shows that the critical depth will occur. For frictionless flow, this equation becomes

$$\frac{dy}{dx} = \frac{S_0}{1 - \alpha\dfrac{bQ^2}{gA^3}} \tag{9.11}$$

Application of Eq. (9.11) will show that the surface curve passes through the critical depth at the crest with a finite slope because

$S = 0$ and $1 - \alpha\frac{bQ^2}{gA^3} = 0$ simultaneously. Starting a short distance on each side of the critical depth, Eq. (9.11) will also show whether the lower or higher stage of flow is to be selected for the upstream and downstream branches of the curve.

The solution assuming frictionless flow is a first approximation to the actual solution. After the approximate water surface is known, the approximate energy line may be drawn by computing the friction loss in short sections or by graphical solution of the equation

$$h_L = \int_0^L \frac{V^2}{C^2R}dx$$

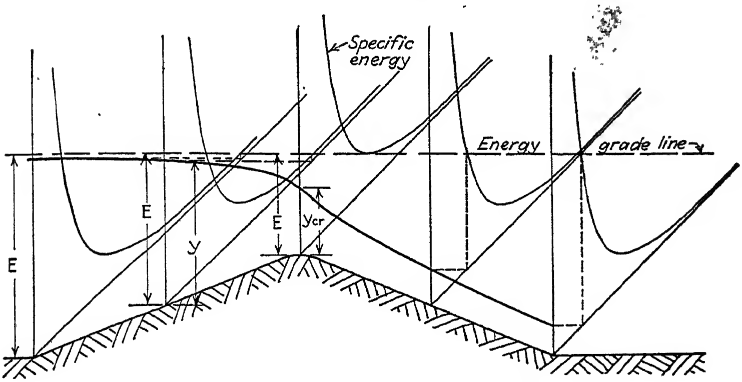


FIG. 147.—Illustrating the use of specific energy curves.

By computing V^2/C^2R for different points along the surface and plotting against x, the planimetered area up to any x gives the loss to that point. The velocities given by the frictionless solution will be too high and therefore the first energy line will lie too low. Using this line, new Q-curves can be constructed, using as E the vertical distance between the bottom and the energy line at that point.

Case II. The Rate of Discharge Is Known.—For the known rate of discharge draw the specific energy curves as shown, by rotating the curves about the 45-deg. line representing the depth. At the crest the elevation of the minimum point on the E-curve has a greater elevation than any of the other minima. Consequently, the elevation of the energy grade line cannot lie below

this elevation with the specified discharge. The two possible depths at the other points are fixed by the intersection of the energy line with the specific energy curves.

Using the curves of specific energy, the second and successive approximations to include the effect of friction do not require redrawing the curves. The approximate energy line is obtained from the depths and velocities for frictionless flow, and a second approximation to the depth is obtained from the intersection with the specific energy curves.

FIG. 148.—Parshall flume on the Fort Lyons Canal, Colorado. Discharge, 1400 cu. ft. per sec. (*R. L. Parshall, "Parshall Flumes of Large Size," Colorado Agr. Exp. Sta. Bull.* 386.)

Interchange of the methods of solution between Cases I and II will illustrate their fields of application. The specific energy curves cannot be drawn unless Q is known while the Q-curves cannot be drawn unless E is known and so the choice of method should depend on whether Q or E is specified.

In channels of constant form, slope, and size, the surface curve may be traced easily by integration of Eq. (9.3) but sharp changes in slope or in form are best treated by means of the Q-curves or curves of specific energy with proper allowances for shock losses as well as friction. Figure 148 shows a large Parshall flume which creates a control section by convergence of the side walls. The flow conditions in the upstream portion of this and similar structures may be analyzed approximately by these methods.

131. Channels of Large Slope or Vertical Curvature.—There is an important group of open-channel problems in which the slope of the bottom is not small. The treatment which follows is based on the work of Lauffer. In Fig. 149, y' is the vertical depth and y the depth perpendicular to the bottom in uniform flow at velocity V. Considering a length L, the weight per unit width is wLy, and this weight is in equilibrium under the action

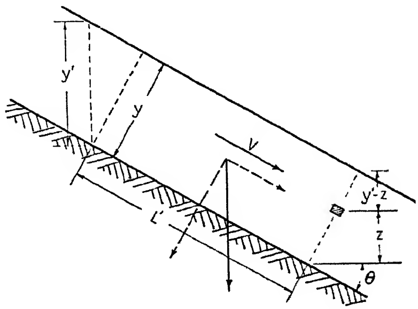


FIG. 149.—Flow in a steeply sloping channel.

of the normal pressure force of the bottom and the tangential force of friction. The equilibrium equation is

$$pL = wLy \cos \theta = wLy' \cos^2 \theta$$

$$\frac{p}{w} = \text{pressure head} = y' \cos^2 \theta$$

$$= y \cos \theta$$

The same line of reasoning would apply at any vertical distance z above the bottom since

$$(y' - z) \cos \theta + z = y \cos \theta$$

Consequently, the summation of the pressure head and the elevation above the bottom, or any other horizontal reference plane, is the same at all layers. The specific energy at any section is then

$$E = y \cos \theta + \alpha \frac{V^2}{2g} = y' \cos^2 \theta + \alpha \frac{V^2}{2g} \tag{9.12}$$

At a slope of 30 deg., $\cos \theta = 0.87$, $\cos^2 \theta = 0.76$, and the error involved is 13 per cent using the normal depth, and 24 per cent using the vertical depth. It is evident that the effect of slope is

important in tracing the water surface down a steep slope. Equation (9.12) applies when accelerations perpendicular to the bottom are negligible. The method of drawing Q-curves and curves of specific energy for steep slopes is evident from Eq. (9.12). Momentum diagrams may be drawn but they appear to be of limited utility for reasons stated previously.

The equation for steady, nonuniform flow in channels of large slope may be derived in the same manner as Eq. (9.3). The total-energy equation is

$$H = z + y \cos \theta + \alpha \frac{V^2}{2g}$$

$$dH = dz + dy \cos \theta + \alpha \frac{V\,dV}{g} = -\frac{V^2}{C^2R}dx$$

Following through the same substitutions as on p. 274 gives

$$\frac{dy}{dx} = \frac{S_0 - \dfrac{Q^2}{C^2RA^2}}{\cos \theta - \alpha \dfrac{bQ^2}{gA^3}} \tag{9.13}$$

Equation (9.13) may be integrated graphically over sections in which S_0 and $\cos \theta$ are constant.

Appreciable vertical curvature brings in the effect of centrifugal forces as well as the effect of slope. The curvature induces a redistribution of pressure and velocity which cannot be predicted by the usual methods of analysis. If the bottom is convex upward, the pressure is less than $y \cos \theta$, and if concave, greater. The effect of vertical curvature on the depth and velocity extends for some distance upstream as well as downstream.

Field data on coefficients of friction applicable to flow in steep channels are very meager but since the velocities are high and turbulence quite fully developed, the normal coefficients should apply. A paper by Lane citing experiments on a steep chute in the Uncompahgre irrigation district tends to substantiate this conclusion.

132. Vertical Drop.—The conditions existing when a channel of small slope is terminated by a vertical drop are of interest in connection with vertical curvature and help to explain the observed characteristics of broad-crested weirs. A great many are possible and so it will be assumed here that the

underside of the jet is fully aerated, that the sides of the channel are vertical, and are continued beyond the drop to prevent lateral spreading. The pressure on the bottom of the channel for some distance upstream must be less than that corresponding to the depth because at the end the bottom pressure is atmospheric. Within the jet the pressure exceeds atmospheric by a small amount because of the convergence of the stream paths, but the jet as a unit is falling freely with a vertical acceleration equal to g. Upstream the vertical acceleration decreases and at each section is a maximum at the surface. The inequality 9.5 appears to indicate that the critical depth should occur at the brink of a vertical drop but examination of the steps

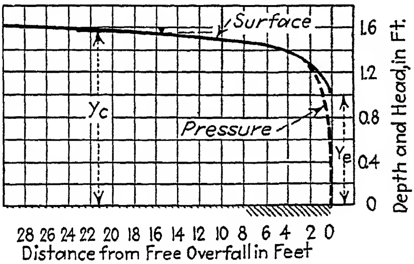


FIG. 150.—Water-surface profile and bottom pressures as measured near the end of a horizontal channel with a free fall at the end. The point of critical depth occurs a distance of 18 times the end depth back of the overfall point, or 11.6 times the critical depth. (*Eng. News-Record*, Sept. 15, 1932, p. 314.)

leading to the equation for critical depth shows that this inequality does not apply if the pressure head at the bottom differs from the depth. In channels of steep slope, the critical depth is obtained from

$$\cos\theta = \alpha\frac{bQ^2}{gA^3}$$

$$\frac{A^3\cos\theta}{b} = \frac{\alpha Q^2}{g} \tag{9.14}$$

If the critical depth is computed from Eq. 9.14 for the channel approaching the drop, it will be found that this depth occurs an appreciable distance upstream. Figure 150 shows the results of measurement of the surface curve near the end of a horizontal channel.

If, in place of a vertical drop, the bottom slope changes gradually, the vertical accelerations are decreased and the dis-

tance upstream to the point of critical depth should decrease. However, it might be even greater than for a drop with complete aeration if the conditions are such as to produce negative pressures.

133. Broad-crested Weirs.—The term *broad-crested weir* is used to designate almost any overfall structure in which the nappe remains continuously in contact with the bottom or is intended to do so. The effects of slope, curvature, and aeration are involved and the conditions are entirely different from those characterizing the sharp-crested weir. Figures 146 and 147 and the accompanying explanation present the basic theory of this type of structure. The critical depth occurs over the crest and the theoretical rate of discharge is

$$Q = 3.09bE^{\frac{3}{2}} \tag{9.15}$$

Actually, broad-crested weirs have coefficients which differ materially from 3.09 and this fact has been pointed to as invalidating the whole analysis. However, when one considers the effect of slope, curvature, and friction on the location of the critical depth and then examines the general form of broad-crested weirs, there appears to be little reason to expect that Eq. (9.15) will apply. The critical depth as computed for channels of small slope has no particular significance under such conditions and there is no reason why it should occur at any predictable point or remain constant in position at different discharges.

The quantity E in Eq. (9.15) is the elevation of the energy grade line above the crest and equals $H + \alpha\frac{V^2}{2g}$, where H is the height of the water surface measured above the same point. The quantity V is a function of the shape of the channel, height of weir, and discharge and, just as in the case of sharp-crested weirs, its inclusion in the equation defining the coefficient does not result in a constant coefficient and needlessly adds to the computations required. Replacing E by H and 3.09 by an empirical coefficient K gives the most convenient form as

$$Q = KbH^{\frac{3}{2}} \tag{9.16}$$

The coefficient K depends upon the geometry of the weir and approach channel and on the Reynolds number, roughness, and,

at low heads, on the surface tension. Since H and V are related, the Reynolds number can be represented in terms of only the head and viscosity. A great many measurements have been made to determine the coefficient K or its equivalent in the equation which includes the velocity head. A very comprehensive series of experiments was made at Cornell University and reported in U. S. Geological Survey *Water Supply Paper* 200.

Variations in the coefficient K are indicated by the following values taken from a table prepared by King.

VALUES OF K FOR BROAD-CRESTED WEIRS

Head, ft.	Breadth of crest, ft.			
	0.5	1 0	3.0	10.0
0.2	2.80	2.69	2.44	2.49
1.0	3.23	2 98	2.65	2 68
5.0	3.32	3.32	3.32	2.64

Attempts have been made to develop a form of broad-crested weir that would cause the critical depth to occur at a single section at all rates of discharge. The supposed advantage of such a device is that the discharge can be computed from the equation for the critical depth. Whether or not such a design can be obtained, it appears to be undesirable because the depth measurement is made at a point at which the surface slope is greater than upstream and the accuracy of measurement is therefore reduced.

The effect of backwater or submergence on broad-crested weirs depends somewhat upon their shape but the general effect is the same for all types. If the downstream water level is increased step by step while the discharge is maintained constant, it is found experimentally that the coefficient as computed from Eq. (9.16) remains constant until the downstream depth measured above the crest is as much as $\frac{2}{3}H$, *i.e.*, when the submergence is very nearly the critical depth. The reason that the critical submergence may even reach the critical depth is that each increment advances upstream as a surge until it reaches water having an equal velocity downstream where its advance is stopped. The next elements traveling over deeper water of

lessened velocity advance a little farther until finally they begin to pass through the control point. The reason that the effect of submergence develops gradually rather than abruptly and that the critical submergence is not equal to the critical depth is to be found in the pressure distribution on the face of the weir. If the pressure on the downstream side of the crest is reduced materially because of curvature or slope, the rising submergence level increases these pressures and reduces the velocities and this effect travels upstream almost instantaneously. One would expect therefore that the critical submergence would be greater for broad weirs of gradual curvature than for other forms and this conclusion is substantiated by the experimental data. Figure 151 shows a typical submergence curve.

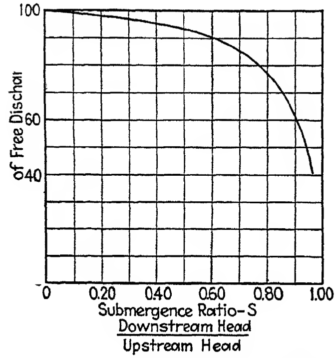


FIG. 151.—Effect of submergence on discharge coefficient for a typical broad-crested weir. (*Data of U. S. Deep Waterways Board.*)

UNSTEADY FLOW

Almost all open-channel flows exhibit variations in surface level which may be periodic, fluctuating with no regular period, or transient. Flood and tidal waves in rivers, seiches in lakes, harbors, weir boxes, and all open bodies of water, and oscillatory waves are commonly observed examples of unsteady flow. The theoretical and experimental treatment of most of these problems has not been reduced to terms such as to be suitable for inclusion here but a few elementary examples will be given to illustrate some of the principles.

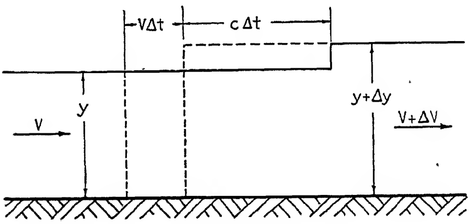


FIG. 152.

134. Positive Surge in a Rectangular Channel.—Flow occurs in a horizontal channel with a velocity V and at a depth y. By means of a gate or other structure located downstream, the velocity is suddenly reduced and the problem is to find out how rapidly this transient condition is transmitted upstream and what increase in depth will result.

Neglecting friction, the momentum equation for a region between (1) and (2) written relative to the wave front is for a channel of unit width:

$$\frac{w}{2}[y^2 - (y + \Delta y)^2] = \frac{w}{g}y(V + c)\Delta V$$

As $\Delta y \rightarrow 0$,

$$dy = -\frac{V + c}{g}dV \tag{9.17}$$

The continuity equation is

$$Vy - (V + \Delta V)(y + \Delta y) = c\Delta y$$

As $\Delta y \rightarrow 0$,

$$dy = -\frac{y}{V + c}dV \tag{9.18}$$

Combining Eqs. (9.17) and (9.18)

$$(V + c)^2 = gy, \qquad c = \sqrt{gy} - V \tag{9.19}$$

In these equations c is the velocity of a wave of small amplitude relative to fixed boundaries. The velocity relative to the water is obtained by making $V = 0$. The resulting value of c equals the critical velocity occurring at critical depth.

The characteristics of a surge of finite height are obtained by assuming it to be made up of an infinite number of infinitesimal waves. Steps in obtaining the differential equation are

$$dy = -\frac{V + c}{g}dV$$

$$V + c = \sqrt{gy}$$

$$\frac{dy}{\sqrt{y}} = -\frac{1}{\sqrt{g}}dV$$

Integrating from y_1 to y_2 and solving for y_2,

$$y_2 = \left[y_1^{\frac{1}{2}} - \frac{1}{2\sqrt{g}}(V_2 - V_1) \right]^2$$

The velocity of the upper layers of a positive surge is greater than that of the lower layers and the upper layers overrun the lower, resulting ultimately in a tumbling motion, and the wave as a whole has a fairly definite and enduring front. The velocity of this wave front is obtained from the continuity equation:

$$c_{AV}(y_2 - y_1) = V_1 y_1 - V_2 y_2$$

$$c_{AV} = \frac{V_1 y_1 - V_2 y_2}{y_2 - y_1}$$

Considering the particular case of a channel in which a gate is closed abruptly, making $V_2 = 0$,

$$\left. \begin{aligned} y_2 &= \left[y_1^{\frac{1}{2}} + \frac{1}{2\sqrt{g}} V_1 \right]^2 \\ c_{AV} &= \frac{V_1 y_1}{y_2 - y_1} \end{aligned} \right\} \tag{9.20}$$

Equations (9.20) were investigated experimentally with the following results (*Transactions American Geophysical Union*, p. 393, 1932):

V_1, ft./sec.	y, ft.	Velocity of surge		Difference, per cent
		Measured, ft./sec.	Computed, ft./sec.	
1.86	0.373	3.26	3.06	−6.1
1.75	0.374	3.20	3.10	−3.1
1.55	0.351	3.00	3.02	+0.6
1.59	0.351	3.00	2.85	−5.0
1.45	0.314	2.89	2.86	−1.0
1.32	0.276	2.68	2.68	0.0

If c_{AV} equals V_1, the wave is stationary and the phenomenon reduces to the hydraulic jump. This condition is shown by the equations to be possible only when $V_1 > \sqrt{gy_1}$.

The conditions assumed in the preceding analysis are much simpler than are usually found in the field. Had the lower gate

been only partially closed, it would have been necessary to compute V_2 corresponding to the new depth y_2, using the discharge equation of the gate. If the gate had been located at the upstream end, closure would have produced a negative wave, or wave of decreased depth, and this wave would have become elongated, since the last elements would travel more slowly than the first.

135. Seiches.—The term *seiche* was used originally to designate the periodic surging in lakes and large bodies of water. For lack of a better term it will be applied to the same phenomenon in any open body of water.

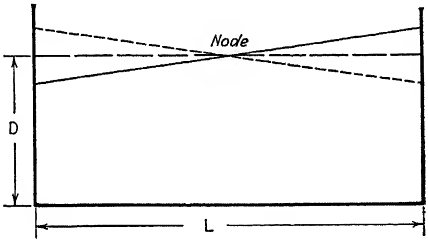


Fig. 153.

Equation (9.19), with $V = 0$, may be used to compute the natural period of oscillation of stationary bodies of water if the amplitude of the oscillation is small as compared with the depth. Referring to Fig. 153, the fundamental period of oscillation is determined by the time necessary to traverse the length twice because at such periods the whole configuration of the surface will be repeated. The period is therefore

$$T = \frac{2L}{\sqrt{gD}}$$

Harmonics of this fundamental period result from the formation of additional nodes. If the depth is irregular in the direction of wave motion, but approximately constant along transverse sections, the fundamental period may be computed from

$$T = 2\int^{L} \frac{dx}{\sqrt{gD_x}}$$

The most widely accepted theory of the tide is that of Harris, who assumed that the phenomenon is due to oscillations between

the continents, and his computations of tidal periods and amplitudes by these equations agree well with the observed values.

If one end of the basin in Fig. 153 were open to a large body of water at constant elevation, the conditions in the basin would correspond to one half of the closed basin and the fundamental period of oscillation would be for constant depth

$$T = \frac{4L'}{\sqrt{aD}}$$

Here L' is the actual length of the open-end basin. If the large body of water oscillates periodically and the period of the connected basin is an harmonic of this forcing frequency, the amplitude at the closed end may become very great. The tides in the Bay of Fundy are an example of this phenomenon.

The theory of seiches is very similar to that of water hammer or acoustic waves in pipes and, in fact, the theories of the two phenomena might well have been treated together. Reflection coefficients are obtained in the same way and the principle of superposition may be applied so long as the total amplitude of the seiche is small as compared with the depth.

136. Oscillatory Waves.—The tractive force of wind exerted on the surface of an open body of water tends to produce waves which differ fundamentally in character from surges. The water particles oscillate in circular or elliptical orbits and the amplitude of their oscillation decreases rapidly with depth below the surface. The subject has been ably summarized by Gaillard who concluded that the so-called *trochoidal theory* agreed best with field measurements. A few of the equations resulting from this theory may be of interest.

It has been found from both theory and observation that the amplitude of the oscillations decreases very rapidly below the surface, and the theory indicates that it becomes negligible at a depth equal to one-half the wave length. Actually, very high waves may cause appreciable movement at greater depths but for purposes of classification, deep-water waves are defined as waves moving in water having a depth greater than one-half the wave length. The wave form is not even approximately sinusoidal and the mean elevation of the water surface does not lie half way between the trough and crest. Considering one wave length, half the energy is in potential form and half kinetic.

Only half of the energy per wave length is transmitted ahead with the wave.

1. *Deep-water Waves.*—The equations defining the characteristics of deep-water waves given by the trochoidal theory are the following:

Wave velocity: $c = \sqrt{\dfrac{gL}{2\pi}} = 2.27\sqrt{L}$ (ft. per sec.)

Radius of orbit (circular): $r = r_s e^{-\frac{2\pi d'}{L}}$ (ft.)

Surface orbit (circular): $r_s = \dfrac{h}{2}$ (ft.)

Still-water level: $a = \dfrac{h}{2} - 0.78\dfrac{h^2}{L}$ (ft.)

$$b = \frac{h}{2} + 0.78\frac{h^2}{L} \text{ (ft.)}$$

Energy per wave length per foot of crest:

$$E = \frac{wLh^2}{8}\left(1 - 4.93\frac{h^2}{L^2}\right) \text{ (ft.-lb. per ft.)}$$

Power transmitted per foot of crest:

$$P = \frac{w\sqrt{L}h^2}{7.05}\left(1 - 4.93\frac{h^2}{L}\right) \text{ (ft.-lb. per sec. per ft.)}$$

where L = wave length.
h = height from trough to crest.
d' = depth below surface.
a = distance from trough to still-water level.
b = distance from crest to still-water level.
r_s = radius of orbit at surface.

The period, length, and velocity of any type of wave are related. The equation is

$$p = \frac{L}{c}$$

The wave theory relates L and c, and theoretically the length and velocity in deep water may both be estimated by observing the period

$$L = 5.12p^2 \text{ (L in feet, p in seconds)}$$
$$c = 5.12p$$

the continents, and his computations of tidal periods and amplitudes by these equations agree well with the observed values.

If one end of the basin in Fig. 153 were open to a large body of water at constant elevation, the conditions in the basin would correspond to one half of the closed basin and the fundamental period of oscillation would be for constant depth

$$T = \frac{4L'}{\sqrt{gD}}$$

Here L' is the actual length of the open-end basin. If the large body of water oscillates periodically and the period of the connected basin is an harmonic of this forcing frequency, the amplitude at the closed end may become very great. The tides in the Bay of Fundy are an example of this phenomenon.

The theory of seiches is very similar to that of water hammer or acoustic waves in pipes and, in fact, the theories of the two phenomena might well have been treated together. Reflection coefficients are obtained in the same way and the principle of superposition may be applied so long as the total amplitude of the seiche is small as compared with the depth.

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Only half of the energy per wave length is transmitted ahead with the wave.

1. *Deep-water Waves.*—The equations defining the characteristics of deep-water waves given by the trochoidal theory are the following:

Wave velocity: $c = \sqrt{\dfrac{gL}{2\pi}} = 2.27\sqrt{L}$ (ft. per sec.)

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Still-water level: $a = \dfrac{h}{2} - 0.78\dfrac{h^2}{L}$ (ft.)

$$b = \frac{h}{2} + 0.78\frac{h^2}{L} \text{ (ft.)}$$

Energy per wave length per foot of crest:

$$E = \frac{wLh^2}{8}\left(1 - 4.93\frac{h^2}{L^2}\right) \text{ (ft.-lb. per ft.)}$$

Power transmitted per foot of crest:

$$P = \frac{w\sqrt{L}h^2}{7.05}\left(1 - 4.93\frac{h^2}{L}\right) \text{ (ft.-lb. per sec. per ft.)}$$

where L = wave length.
h = height from trough to crest.
d' = depth below surface.
a = distance from trough to still-water level.
b = distance from crest to still-water level.
r_s = radius of orbit at surface.

The period, length, and velocity of any type of wave are related. The equation is

$$p = \frac{L}{c}$$

The wave theory relates L and c, and theoretically the length and velocity in deep water may both be estimated by observing the period

$$L = 5.12p^2 \text{ (L in feet, p in seconds)}$$
$$c = 5.12p$$

2. *Shallow-water Waves.*—As waves run into shallow water their velocity and length decrease but if they remain intact their period must remain the same. The orbits followed by the water particles become elliptical and the height increases. The equations given by the trochoidal theory are the following:

Surface orbit: $a_s = b_s \dfrac{e^{\frac{4\pi d_0}{L}} + 1}{e^{\frac{4\pi d_0}{L}} - 1}$ (ft.)

Height: $h = 2b_s$, ft.

Wave velocity: $c = 2.27\sqrt{\dfrac{b_s}{a_s}L}$ (ft. per sec.)

Energy per wave length per foot of crest:

$$E = \frac{wLh^2}{8}\left(1 - 19.7\frac{a_s^2}{L^2}\right) \text{ (ft.-lb. per ft.)}$$

Power transmitted:

$$P = \frac{wh^2\sqrt{\dfrac{b_s}{a_s}L}}{7.05}\left(1 - 19.7\frac{a_s^2}{L^2}\right) \text{ (ft.-lb. per sec. per ft.)}$$

where a_s = semimajor axis (horizontal) of surface particles.
b_s = semiminor axis (vertical) of surface particles.
d_0 = depth of water.

The ratio a_s/L is a function of d_0/L. The numerical values are as follows:

d_0/L	0.10	0.12	0.16	0.20	0.24	0.30	0.33
h/a_s	1.10	1.27	1.53	1.70	1.81	1.91	1.94

If the depth and wave length are known, the power transmitted and energy per wave length may be computed by replacing a_s by h through the medium of the table. If the height and period of a wave are observed in deep water, the energy per wave length may be computed from the equations given. The height and length in shallow water at any specified depth may then be computed approximately by trial and error, assuming that the energy per wave length remains constant. The height increases during shoreward motion and finally the wave *breaks* when the height becomes too great in proportion to the depth. This

critical height appears not to be a function of depth alone but of the wave length and slope of bottom as well.

Since an oscillatory wave has a certain amount of energy per wave length, an increase of crest length should result in a decrease in height which can be computed approximately from

$$bE = \text{const.}$$

Here, b is the length along the crest and E is the energy per wave length per foot of crest.

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Problems

1. A dredged earth channel of trapezoidal section has a bottom width of 30 ft. and side slopes at an angle of 1 vertical to 3 horizontal. Flow occurs at a depth of 14 ft. If the discharge is 2000 cu. ft. per sec., what must be the slope of the energy gradient? *Ans.* 0.0000926.

2. A concrete trapezoidal canal has a bottom width of 5 ft. and side slopes of 1:1. If the slope is 0.001, compute the discharge during uniform steady flow at a depth of 5 ft. *Ans.* 296 c.f.s.

3. A rectangular open channel has a bottom width of 8 ft. and a surface roughness corresponding to $n = 0.015$. If the slope of the bottom is 0.001 and the depth of flow 4 ft., what is the discharge under conditions of uniform steady flow? Compute depth of flow in a channel of the same surface

roughness but triangular in section with a 90-deg. angle between the sides if the slope and quantity are to be the same as for the above rectangular section. (Use Manning formula.)

Ans. (*a*) 159.3 cu. ft. per sec.; (*b*) 5.66 ft.

4. A semicircular open channel is carrying a uniform flow of 350 cu. ft. per sec. The character of the surface is such that the coefficient in the Chezy formula is 125. If the slope at the bottom is 0.002 and the channel is flowing full what is the diameter of the channel?

5. A trapezoidal channel has a base width of 6 ft., side slopes of 1 vertical to 2 horizontal, a bottom slope of 0.0002 and a surface roughness indicated by $n = 0.022$ in Manning's formula. If the depth of flow is 4.3 ft., find the rate of discharge of water in cu. ft. per sec.

6. A dredged earth channel, with a base 10 ft. wide and side slopes making an angle of 30 deg. with the horizontal, carries water with a depth of 4 ft. If the channel is to carry 300 cu. ft. per sec., what is the required slope when uniform steady motion is assumed? What is the value of C for this channel?

Ans. (*a*) 0.00226; (*b*) 57.6.

7. What is the effect on the nonuniform-flow equation [(9.3), page 274] if acceleration is neglected in its derivation?

8. On the basis of Manning's formula, what is the relative carrying capacity of three rectangular open channels, each having an area of 16 sq. ft., but having dimensions as follows: (*a*) 2 ft. wide by 8 ft. deep, (*b*) 4 ft. wide by 4 ft. deep, (*c*) 8 ft. wide by 2 ft. deep.

Ans. (*a*) 0.764; (*b*) 1.000; (*c*) 1.000.

9. A rectangular concrete channel 3 ft. wide is to carry 30 cu. ft. per sec. The slope of the bottom is 0.001. Compute the velocity and the depth of flow by Manning's formula. *Ans.* (*a*) 3.16 ft. per sec.; (*b*) 3.16 ft.

10. Water is flowing in a wide channel at a depth of 6 ft. The slope of the channel bottom is 0.0006 and C in the Chezy formula is 79. A sharp-crested weir 7.5 ft. high is placed across the channel. Compute and plot the backwater curve for the channel. Use $Q = 3.33\ LH^{\frac{3}{2}}$ for weir discharge.

11. Water flows at a depth of 2.0 ft. and a velocity of 40 ft. per sec. in a trapezoidal channel having a bottom width of 4 ft. and side slopes of 1 vertical to 2 horizontal. Calculate (*a*) the depth necessary to produce the hydraulic jump by means of the momentum curve, (*b*) the critical depth, (*c*) the energy loss.

12. The channel of Prob. 10 has an abrupt drop at the end. How far from the drop will the depth of flow be (*a*) 5.0 ft., (*b*) 5.7 ft.?

13. Water flows in a rectangular channel 4 ft. wide. If the bottom is horizontal and the depth is 2.5 ft. when the flow is 30 cu. ft. per sec., what are the height and velocity of the wave produced when the flow is suddenly obstructed causing a 50 per cent decrease in the channel velocity? Determine the wave height and velocity if the obstruction were to produce complete stoppage of flow.

Ans. (*a*) 7.43 ft. per sec.; (*b*) 0.89 ft.; (*c*) 8.43 ft. per sec.

14. Water flows in a rectangular channel with a horizontal bottom to a depth of 5 ft. If the initial velocity of 8 ft. per sec. is suddenly reduced 40 per cent, how far upstream from the obstruction will the wave travel during the first 8 sec. following the velocity reduction?

CHAPTER X

MODELS OF OPEN-CHANNEL FLOW

The conditions for similarity of submerged flow have been discussed in detail in Chap. IV. There the Reynolds number was found to be the principal criterion for dynamical similarity of flow under the action of inertia and viscous forces without appreciable gravitational effects and is, therefore, the model law for the study of airfoils, venturi meters, centrifugal pumps, pipe friction, and all of the other flow phenomena in which the working fluid is confined within solid boundaries.

In Chap. V another model law was mentioned briefly, namely, the conditions for similarity under the action of inertia and surface tension, and in Chap. VI the basic relationships in testing ship models were discussed (Froude's model law).

The present chapter deals with the use of models of open-channel structures and natural water courses and is particularly concerned with those problems which are ordinarily classified as falling in the field of civil engineering. The majority of models of this class represent flow systems which cannot be treated with sufficient accuracy by present analytical methods.

137. Definition of a Model.—There is a tendency to think of models and model laws in terms of the dictionary definition of a model as a small-scale reproduction of a larger object or prototype and this conception tends to limit their use. In the hydraulic sense, a model is a flow system which may be used to predict the characteristics of another flow system of different absolute size or under different conditions. Tests of a venturi meter or orifice with air to predict its characteristics with water are model experiments in this sense even though no change in size is involved. Models of very small flow systems, such as carburetor jets, might well be several times the size of the prototype. Furthermore, it is by no means necessary that a model *look* like the prototype or that all its dimensions be reduced to a single scale. For example, a photograph of a model of a pipeline and surge chamber might bear almost no resemblance to the proto-

type since the flow is essentially unidirectional and the piping may be arranged in any convenient manner so long as the friction and inertia are correctly represented.

From a slightly different point of view a model may be regarded as a mechanical means of solving the differential equations of fluid motion for a given set of boundary conditions, the essential requirement being that it be possible to establish quantitative rules for transferring data. The term *boundary condition* is used here in the mathematical sense and includes the distribution of velocity and pressure over the ends of the model as well as the form of the solid boundaries. In general, strict geometrical similarity is convenient but departures from it are frequently necessary both in the over-all dimensions and particularly in the matter of roughness.

138. References.—So much work is now being done with models that mention of a few sources of information may prove useful. In 1926 a description of the principal European laboratories and their work was published in Germany and an extension of this work to include some of the American laboratories appeared in 1929 under the title "Hydraulic Laboratory Practice." More recent work, particularly on the reliability of models, was presented at the Navigation Congress at Venice. Current work in the United States is described in a bulletin published semiannually by the Bureau of Standards. Articles dealing with details of theory and descriptions of particular models have appeared frequently in recent years in the technical journals. These sources of information are mentioned because experience in model design is important and one can benefit materially from descriptions of models used under similar circumstances.

GEOMETRICALLY SIMILAR MODELS

139. Model Law for Gravity Forces (Froude's Model Law).—In Chap. IV the conditions for the dynamical similarity of geometrically similar flow systems were obtained from a consideration of the forces necessary to move the water particles along similar paths. In an open channel, the depth, surface slope, and other features of the flow are controlled by the joint effect of inertia and the gravitational force. The ratio of the inertial reactions of similar volumes similarly located is

$$b_f = \frac{MA}{ma} = b_m b_l b_t^{-2} = b_\rho b_l^4 b_t^{-2} \tag{10.1}$$

Here, the b indicates a ratio of the quantity appearing as a subscript and the capital letters refer to the prototype. The numerical value of the scale ratio is equal to a magnitude in the prototype divided by the corresponding magnitude in the model. If an equation is dimensionally homogeneous, the model laws may be obtained directly by substituting scale ratios for the quantities in the equation since this will evidently be the final

FIG. 154.—Model of a hydroelectric development at Cedar Falls, Wis. Scale 1:96. (*Courtesy Lorenz G. Straub.*)

result of dividing the equation for the prototype by that for the model. For complete similarity of flow, the ratios of all kinds of forces operative in the model must equal the ratio expressed by Eq. 10.1. The ratio of the gravitational forces is

$$\frac{MG}{mg} = \frac{\rho L^3 G}{\rho_m l^3 g} = b_\rho b_g b_l^3 \tag{10.2}$$

Usually $b_g = 1$, since the variation in the gravitational constant is small, but the possibility of artificially varying the effective value of b_g should not be overlooked. The conditions for similarity are then that

$$\begin{aligned} b_\rho b_l^4 b_t^{-2} &= b_\rho b_g b_l^3 \\ b_l b_t^{-2} &= b_g \end{aligned} \tag{10.3}$$

Equation (10.3), which is the law of Froude, is a kinematical relationship and states the conditions for similarity of flow of any liquid, regardless of density. Hence, the density of the

liquid used in the model is immaterial. Since water is plentiful it is generally used in models of open-channel flow, but any other liquid of small viscosity might be used.

For $b_g = 1$, the model ratios become, in terms of the linear scale ratio,

Time: $b_t = b_l^{\frac{1}{2}}$
Velocity: $b_V = b_l^{\frac{1}{2}}$
Rate of discharge: $b_Q = b_l^2 b_v = b_l^{\frac{5}{2}}$
Acceleration: $b_a = b_l b_t^{-2} = 1$
Force: $b_f = b_\rho b_l^3$
Static pressure: $b_p = b_w b_l = b_\rho b_l$
Dynamic pressure: $b_p = b_\rho b_v^2 = b_\rho b_l$
Head: $b_h = b_p b_w^{-1} = b_l$
Power: $b_P = b_w b_Q b_h = b_\rho b_l^{\frac{7}{2}}$
Slope: $b_S = 1$

The ratios of dynamical quantities such as force and pressure bring in either the density or the specific weight, the two ratios being identical with $b_g = 1$.

Since a model is a real flow system in which friction plays an important part it is necessary to investigate the effect of viscosity or, in other words, to apply the Reynolds law of similarity which is

$$\frac{b_\rho b_V b_l}{b_\mu} = 1$$

If b_ρ and b_μ are both unity, as for pure water at the same temperature, in model and prototype, the Reynolds law requires that

$$b_l = b_V^{-1}$$
$$b_l^2 = b_t^{-1} \tag{10.4}$$

For Eqs. (10.3) and (10.4) to be satisfied simultaneously with $b_g = 1$, it is necessary that $b_l = 1$ and the model is of the same size as the prototype.

Considering the general case of b_μ and b_ρ not equal to unity, the Froude and Reynolds criteria may be satisfied simultaneously with $b_g = 1$ only if $b_l^{\frac{3}{2}} = b_\mu / b_\rho$. This condition is generally impossible of fulfillment but it is also unnecessary under certain conditions. The statement often made regarding this problem is that the Reynolds criterion need not be satisfied in open-channel flow because friction is of minor importance; but this is

evidently erroneous because friction has been a major factor in at least half of the models built in accordance with the Froude law. Solution of the difficulty lies in the fortunate circumstance that in turbulent flow the friction coefficient is very nearly constant. In fully developed turbulent flow, the friction coefficient is actually constant and independent of the viscosity or Reynolds number. Under such circumstances, the friction losses fulfill the model law since then $h_L = KV^2$, and $b_l = b_V^2$.

Another method of deriving the model laws is to assume frictionless flow and consider the relationship between changes in velocity head and total energy. From Bernoulli's equation,

$$\Delta\left(\frac{V^2}{2g}\right) = \Delta E$$

The quantity E is a linear dimension which for similarity must follow the ratios of the other lengths, hence

$$\frac{b_V^2}{b_g} = b_l = \frac{b_l^2}{b_t^2 b_g}$$

$$b_l b_t^{-2} = b_g$$

which is identical with Eq. (10.3).

The derivation on the basis of dynamical similarity shows that this model law applies to both steady and unsteady flow.

140. Critical Velocities.—The minimum possible velocity of a small capillary wave or ripple at the interface between air and water at normal temperature is approximately 0.76 ft. per sec. If the velocity of the water relative to obstructions such as model piers is less than this minimum velocity, stationary waves do not deform the surface. At some greater velocity, the effect of surface tension becomes negligible and it is necessary that this at present indefinite, critical velocity be exceeded if surface deformation is of importance.

The second critical velocity which must be considered is that separating laminar from turbulent flow. On the basis of pipe experiments, the critical Reynolds number above which the flow is generally turbulent should be about 3000. Remembering that $R = 4D$, the critical Reynolds number based on the hydraulic radius would be

$$\text{Re}' = \frac{VR}{\nu} > 750$$

Authorities disagree on the proper value. The experiments of Allen and the U. S. Waterways Experiment Station show much higher critical values, the exact critical point varying with hydraulic radius and roughness. In both sets of experiments the flow was always turbulent for $\mathrm{Re}' = VR/\nu > 4000$ which corresponds to $\mathrm{Re} = VD/\nu > 16{,}000$ for pipes. To fix the magnitudes in mind, assume $R = 0.3$ ft. and $\nu = 1.0 \times 10^{-5}$ ft.^{2} per sec. The corresponding critical velocity given by Allen is 0.13 ft. per sec. Considering only the transition from laminar to

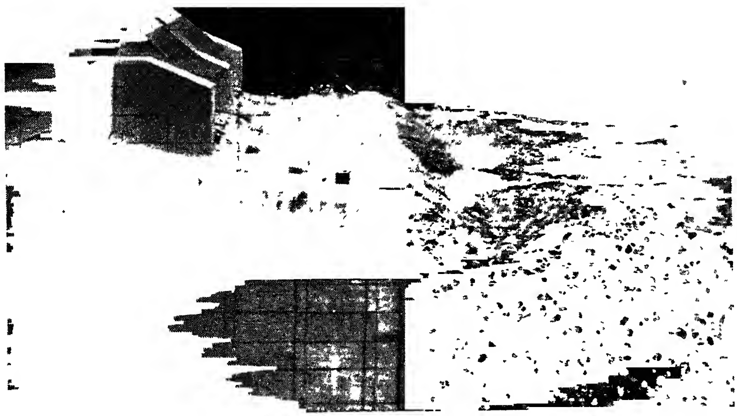

FIG. 155.—Partial model of a tainter gate installation on the Mississippi River at St. Cloud, Minn. Scale 1:24. (*Courtesy Lorenz G. Straub.*)

turbulent flow, $\mathrm{Re}' > 4000$ may be high but it must be remembered that there is a range of Reynolds number between the change from laminar to turbulent flow and the point of fully developed turbulence. The condition for similarity is that $h_L \propto V^2$ and to ensure this condition an even greater lower limit should be set. It should also be recalled that the hydraulic radius is an acceptable representation of hydraulic size only for channels of simple form. If the model represents a river in the *overbank* condition, laminar flow might easily occur in the model in the overbank region even though the Reynolds number for the channel as a whole satisfies the criterion.

In the theory of hydraulic friction and roughness treated in Chap. IV, two critical points have a bearing on the theory of models. von Kármán states that when $V^*k/\nu < 3$ the surface is hydraulically smooth while with $V^*k/\nu > 60$, the surface is hydraulically rough and the friction factor is constant. Here, $V^* = \sqrt{\tau_0/\rho}$, τ_0 is the tractive force at the boundary, k is the average size of granular roughness, and ν is the kinematic viscosity. If the slope of a wide channel is 0.0005 and the depth 3 ft.,

$$\tau_0 = w\,DS = 62.4 \times 3 \times 0.0005 = 0.093 \text{ lb. per sq. ft.}$$

and $V^* = \sqrt{0.093/1.98} = 0.218$ ft. per sec. For fully developed turbulence under these conditions $k > 2.76 \times 10^{-3}$ ft. corresponding to a fairly fine sand. If the same channel were modeled to a scale $b_l = 10$,

$$\tau_{0m} = 62.4 \times \tfrac{3}{10} \times 0.0005 = 0.0093 \text{ lb. per sq. ft.}$$

and for fully developed turbulence, $k > 8.7 \times 10^{-3}$ ft. Instead of a reduction in the size of the roughness, this criterion indicates an increase. The next section will show that this criterion should not be applied and it is to be concluded that in most instances the prototype operates above the point of fully developed turbulence while the model is below it, thus causing some disagreement.

141. Friction Slope.—Since the bottom and water-surface slopes are to be the same in model and prototype, it follows directly that the slopes of the energy grade lines should be equal. Using Manning's equation as the basis for analysis,

$$V = \frac{1.49}{n} R^{\frac{1}{6}} (RS)^{\frac{1}{2}}$$

$$b_V = \frac{b_R^{\frac{1}{6}}}{b_n} (b_R)^{\frac{1}{2}}$$

But $b_R = b_l$ and $b_V = b_l^{\frac{1}{2}}$ so that the condition for equality of slopes is that $b_n = b_l^{\frac{1}{6}}$. For example, if the scale ratio is 100 and n in nature is 0.018, the proper value of n in the model is

$$\frac{0.018}{(100)} = 0.0083$$

which is very near the lower limit of n. Strickler states that the friction coefficient n may be expressed as $n = Ck^{\frac{1}{6}}$. If this is

correct, $b_k = b_l$ for similarity, or in other words, the roughness must be reduced in proportion to the scale of linear dimensions. This conclusion is entirely reasonable but, on the basis of the roughness criterion given by von Kármán, it tends to further reduce the flow in the model below the point of fully developed turbulence.

It has been mentioned previously that the equations for friction losses in open-channel flow probably apply only in the region of fully developed turbulence and it is doubtful if they are directly applicable to models of usual sizes.

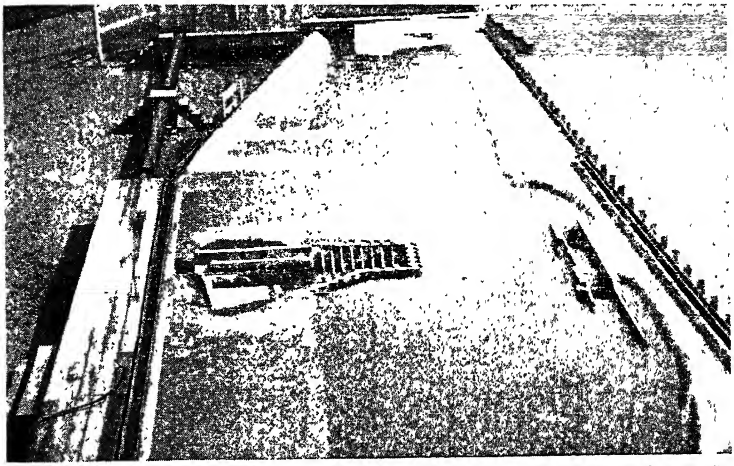

FIG. 156.—Model of Pickwick Landing dam site on the Tennessee River built to study flow conditions during construction. Scale 1:100. (*Courtesy Tennessee Valley Authority.*)

If an undistorted model is constructed with a fixed bed, an approach to the proper value of n may be made experimentally by altering the character of the surface until the proper water-surface slope is obtained at a known discharge. However, if the model is formed in movable materials, the material size cannot usually be controlled so as to give the proper n because of ripple formation, even though the median diameter is reduced to the proper scale.

The discrepancies between model and prototype will be greater for models in which friction is important, such as river models, than in those showing rapid changes in velocity such as spillways, chutes, weirs, and so forth. No method of com-

pensating for the error due to friction has been devised which is comparable with that used for ship models. One method applicable only to relatively straight channels of constant section is to tip the geometrically similar model as a unit so as to compensate for the excessively large friction.

142. Examples of Model Design.—The first step in laying out a model is to prepare a detailed statement of the problems to be investigated and the accuracy required. The size of the model will depend upon the features of the flow which are of interest and the possible accuracy of measurement as well as upon the hydraulic criteria mentioned. One of the first decisions to be made is whether the model is to be constructed with a fixed or movable bed. If lines of symmetry can be expected to occur, one may employ half models or sectional models by substituting solid boundaries for these lines of symmetry. For example, if the problem is to determine the discharge coefficient of spillway gates set between identical piers, a model may be built of one gate with the solid walls of the channel splitting the piers or with one pier in the center and the channel walls along the center line of the adjacent gate openings. A very long weir without piers may be tested by modeling a section of it. Sectional models have been used successfully, but in using them one must remember that each of the walls represents an assumed line of symmetry in the flow. Recalling the staggered system of vortices which appear behind cylinders and flat plates (see Chap. VI), assumptions of this kind are fraught with some dangers. Models must usually be fitted into a particular space and it is sometimes advantageous to make the model a reflected image of the prototype. After these general features have been decided tentatively, the model laws are applied and the various criteria computed.

Examples: 1. The coefficient of discharge of an overflow dam is to be determined by means of a model. The crest length is 30 ft. and the approximate discharge 4000 cu. ft. per sec. at an 11-ft. head. The water supply available is 1 cu. ft. per sec. Determine the maximum scale ratio and the hydraulic quantities in the model. The discharge scale controls.

$$b_Q = \left(\tfrac{4000}{1}\right) = b_l^{\frac{5}{2}}$$
$$b_l = 27.6$$

The model quantities are $Q_m = 1$ and $h_m = 0.40$ ft. For similarity the measured coefficient should be the same in the prototype as in the model.

2. A spillway 20 ft. high and 600 ft. long without piers discharges 16,000 cu. ft. per sec. under approximately a 4-ft. head. The necessary backwater to form an hydraulic jump on the apron is to be determined by means of a model. The laboratory has a channel 12 in. wide, 18 in. deep with a water supply of 1.3 cu. ft. per sec. Proportion the model.

The scale evidently cannot be reduced to accommodate the whole crest length for the head would then be only 0.0067 ft. Conditions point to a sectional model. The discharge per foot of crest is 16,000/600 = 26.7 cu. ft. per sec. and the minimum scale ratio corresponding to the available flow is

$$b_l = b_Q^{\frac{2}{3}} = \left(\frac{26.7}{1.3}\right)^{\frac{2}{3}} = 7.48$$

The total depth for this scale ratio is 3.21 ft. and the depth of the channel controls. Allowing 3 in. of freeboard,

$$b_l = \frac{20 + 4}{1.25} = 19.2 \approx 20$$

The discharge per foot of model crest is

$$q = \frac{16,000}{600} \cdot \frac{1}{b_l^{\frac{3}{2}}} = 0.298 \text{ cu. ft. per sec.}$$

The head on the model is 0.20 ft.

3. A geometrically similar model with fixed bed is to be used to study the main current in a section of river channel having an average width of 2000 ft., a maximum depth of 40 ft., a velocity of 2.5 ft. per sec. and a discharge of 125,000 cu. ft. per sec. The average hydraulic radius is 25 ft. and the approximate value of Manning's n for the reach is about 0.02. The laboratory space available limits the width of the model to 6 ft. Proportion the model and check against the model criteria.

The minimum scale ratio is $\frac{2000}{6} = 334$. Use 324, the square of 18. The Reynolds number in the model assuming water at 70 deg. Fahr. is

$$\frac{VR}{\nu} \cdot \frac{1}{b_l^{\frac{3}{2}}} = \frac{2.5 \times 25}{1.07 \times 10^{-5} \times (324)^{\frac{3}{2}}}$$

This value is dangerously near the lowest recommended value of Re and considerably below that found by Allen to ensure turbulent flow. The model velocity would be 0.14 ft. per sec. and the maximum depth 0.12 ft., if the flow conditions agreed with the model laws. A geometrically similar model built to fit the space available would be unsatisfactory both because of the low Reynolds number and because of the small velocities. A distorted model is indicated.

DISTORTED MODELS

The last example in the preceding section indicated the desirability of increasing the velocities and depths in the model without increasing the horizontal dimensions. Such distortion

of models is permissible under certain conditions. The principal ways in which models are distorted are the following:

1. Distortion of roughness from strict geometrical similarity, including size of sand for movable bed models.
2. Modification of slope by tilting model as a whole or in part.

FIG. 157.—A portion of a model of the Mississippi River between Helena, Ark. and Donaldsonville, La. Horizontal scale 1:2000. Vertical scale 1:110. The entire model is 1050 ft. long and 107 ft. wide. Corrugated screening is used to simulate the roughness of wooded portions of the floodway. (*Courtesy U.S. Waterways Experiment Station.*)

3. Distortion of controllable hydraulic quantities, such as rate of discharge and time.
4. Use of different vertical and horizontal scale ratios.
5. Use of different scale ratios for vertical, transverse, and longitudinal dimensions.
6. Nonsimilarity of form with equivalent hydraulic dimensions.

Several of these types of distortion are usually involved simul-

taneously. The term *distorted model* is generally applied to models such as described in (4), and only this type will be considered in the following sections. Type (5) has been used in the study of the motion of flood waves. The model shown in Fig. 161 includes what is known as a *river labyrinth* which represents the effect of the maritime section of the river above the estuary. This labyrinth has the proper average width, depth, and tidal prism at all river discharges, but it is not geometrically similar and this part of the model falls under type (6) above. The remainder of the model is the part actually under investigation.

143. Surface Profiles.—Considering distorted models constructed with different horizontal and vertical scales, the principal limitation of the method may be defined by means of the equation for nonuniform flow. A convenient form of this equation is

$$\Delta h = Q^2\left(\frac{1}{C^2RA^2} - \frac{2B}{gA^3}\right)\Delta x \tag{10.5}$$

where h = the elevation of the water surface above a horizontal reference plane.

x = a distance along the channel.

Q = the discharge.

C = the Chezy coefficient.

R = the hydraulic radius.

B = the surface width.

A = the area.

Regarding Eq. (10.5) as representing the surface curve in the prototype, the equation for the corresponding section of the model may be obtained by dividing each quantity by the proper scale ratio. Assuming that C is the same in model and prototype and that the hydraulic radius follows the vertical scale ratio,

$$\frac{\Delta h}{b_z} = \frac{Q^2}{b_Q^2}\left(\frac{b_z{}^3b_x{}^2}{C^2RA^2} - \frac{2Bb_z{}^3b_x{}^2}{gA^3}\right)\frac{\Delta x}{b_x} \tag{10.6}$$

Here b_z is the vertical and b_x the horizontal scale ratio. Dividing Eq. (10.5) by Eq. (10.6) and solving for b_Q,

$$b_Q = b_z{}^2b_x{}^{\frac{1}{2}}$$

However, the scale ratio b_Q should be $b_xb_z{}^{\frac{3}{2}}$ to correspond with the model law for distorted models and it appears that models

may not be distorted if the flow is markedly nonuniform. The discrepancy disappears if $b_z = b_x$, giving $b_Q = b_x^{\frac{5}{2}}$, which is the same ratio as given on page 311. In fact the method employed here is a derivation of the model laws for geometrically similar models.

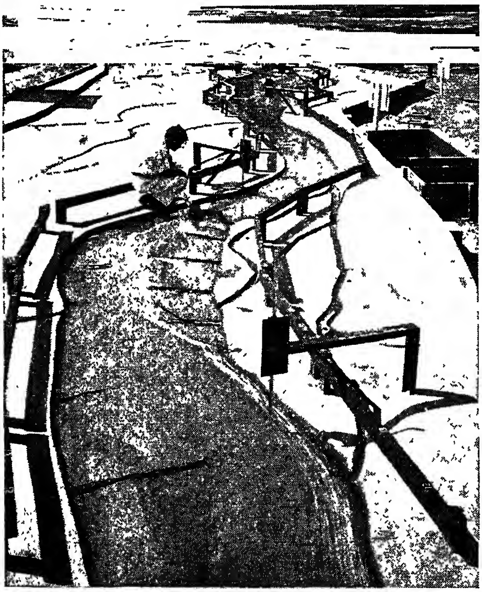

FIG. 158.—Movable bed model of the Mississippi River near Fort Chartres. White streaks show the course of the water at low stages. Horizontal scale 1:1000. Vertical scale 1:125. (*Courtesy U. S. Waterways Experiment Station.*)

The use of distorted models is restricted to situations in which friction is large in comparison with changes in velocity head. The unavoidable error due to distortion becomes small under such circumstances.

144. Model Law.—The usual derivation for *distorted models* proceeds from the relationship

$$\Delta\left(\frac{V^2}{2g}\right) = \Delta E, \qquad b_V{}^2 = b_z$$

The transverse areas at corresponding sections stand in the ratio $b_x b_z$, and the scale ratio of discharges should be $b_Q = b_x b_z^{3/2}$. This derivation is theoretically erroneous because it proceeds from the equation for the change in velocity head during frictionless flow which is exactly the situation in which distorted models should not be used.

Distorted models may be used for studying rivers and other flow systems in which the effect of friction is large in comparison with the change in velocity head and derivation of the model law should recognize this fact. The objective of such experiments is to study the current systems at bends, behind dikes, and so forth, and it is essential that the dynamical conditions be such as to produce similarity of flow paths. The water-surface slope necessary to produce a certain curvature of path is V^2/r where V is the velocity of flow and r is the radius of curvature. The model ratio for such slopes is therefore $b_V{}^2 b_r{}^{-1}$. The ratio of the slope of fixed boundaries in nature to that in the model is $b_z b_x{}^{-1}$ and for similarity this must also be the scale of water-surface slopes. Therefore,

$$b_V{}^2 b_r{}^{-1} = b_z b_x{}^{-1}$$

But the paths of the water particles are predominantly horizontal under the assumed conditions. Therefore

$$b_x = b_r \qquad \text{and} \qquad b_V{}^2 = b_z \tag{10.7}$$

All the other scale ratios are obtained from Eq. (10.7) in the same manner as for geometrically similar models. A few of these ratios are:

Discharge: $b_Q = b_x b_z^{3/2}$
Static pressure: $b_p = b_w b_z$
Dynamic pressure: $b_p = b_\rho b_V{}^2 = b_w b_z$
Time: $b_t = b_x b_V{}^{-1} = b_x b_z^{-1/2}$

Although Eq. (10.7) is identical with the velocity scale ratio obtained from the energy equation, its derivation gives it a very different significance.

145. Friction Losses.—If Eq. (10.7) is to be satisfied, the flow must be turbulent and therefore the Reynolds number must exceed a certain minimum value as mentioned in Sec. 140. The velocity, hydraulic radius, and roughness are to be reproduced in the model in such a manner that the velocity scale is given by

Eq. (10.7). Analysis of the problem is most conveniently made by Manning's equation.

The slope ratio is $b_S = b_z b_x^{-1}$ and the velocity ratio $b_V^2 = b_z$. Manning's equation is

$$V = \frac{1.49}{n} R^{\frac{1}{6}} (RS)^{\frac{1}{2}}$$

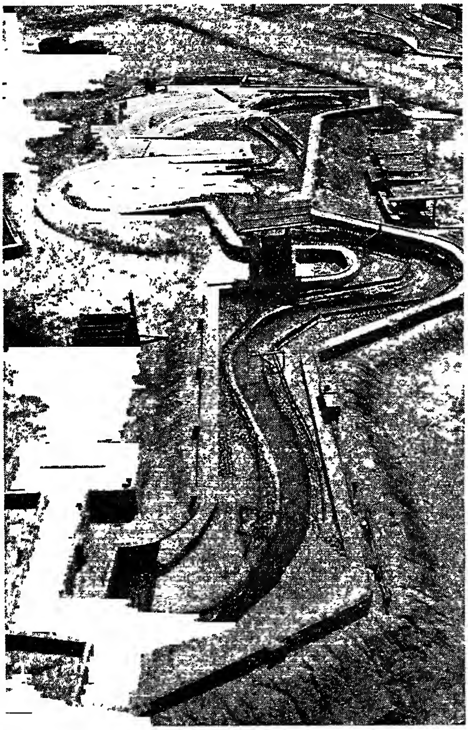

FIG. 159.—Movable bed model of the Savannah River about 186 miles above Savannah, Ga. The roughness of wooded areas is simulated in this model by coarse gravel. (*Courtesy U. S. Waterways Experiment Station.*)

If the model channel is wide in proportion to the depth, the hydraulic radius follows the scale of vertical dimensions and therefore

$$b_V = \frac{b_z^{\frac{2}{3}} b_S^{\frac{1}{2}}}{b_n}$$

Substituting for b_V and b_S,

$$b_z^{\frac{1}{2}} = \frac{b_z^{\frac{2}{3}} b_z \, b_x^{-\frac{1}{2}}}{b_n}$$

$$b_n = b_z^{\frac{2}{3}} b_x^{-\frac{1}{2}} \tag{10.8}$$

For example, if $b_z = 100$ and $b_x = 1000$, then $b_n = 0.68$; *i.e.*, the roughness coefficient of the model is to be 1.47 times greater in the model than in nature.

Fixed-bed models do not present a very great problem in the matter of roughness because they can be roughened locally until the surface curve agrees with the measured curve in nature at a known discharge. However, the choice of bed materials in movable-bed models must be based largely on their characteristics as regards movement, and control of roughness by control of size is not practicable, especially since ripple formation has an unpredictable effect. One method employed to compensate for incorrect roughness is to calibrate the model for its discharge characteristics, *i.e.*, determine the discharge which gives a water-surface slope corresponding to that in nature. The ratio of the corresponding discharge in nature to the one thus found is taken as the discharge scale. The resulting velocity scale is not in accordance with Eq. (10.7) and some discrepancy in the current system is to be expected.

146. Hydraulic Radius and Velocity Distribution.—The hydraulic radius includes both horizontal and vertical dimensions and its scale ratio in a distorted model depends upon the form of the channel. Considering a rectangular channel of width B and depth D, the hydraulic radii in prototype and model are

$$R = \frac{BD}{B + 2D}, \qquad r = \frac{Bb_x^{-1}Db_z^{-1}}{Bb_x^{-1} + 2Db_z^{-1}}$$

$$b_R = \frac{R}{r} = b_x b_z \cdot \frac{Bb_x^{-1} + 2Db_z^{-1}}{B + 2D}$$

The scale ratio depends upon the ratio B/D. If the channel *in the model* is very wide in proportion to the depth, $2D$ is negligible in comparison with B and the ratio becomes $b_R = b_z$. If $B = 100$, $D = 5$, $b_x = 100$, and $b_z = 20$, the hydraulic radii are $R = 4.55$ ft. and $r = 0.167$ ft., and $b_R = 27.3$. In this example with $B/D = 20$, the hydraulic radius in nature differs from the depth by about 10 per cent. Equation (10.8) might be

modified as a second approximation to include the true scale of hydraulic radii but this is not of much assistance because the roughness must be adjusted empirically.

The effect of the ratio of width to depth on the velocity distribution was discussed in Chap. IX. A decrease in this ratio appears to result in a depression of the point of maximum velocity and therefore in a distorted model one may expect to find a change in the velocity distribution which is not in accordance with the scale ratio. The magnitude of the change probably can be expressed in terms of the ratio of hydraulic radius to average depth in nature, and the ratio of horizontal to vertical scales. Data on this effect are meager but certain preliminary results indicate the desirability of distorting models as little as possible because of the change in the velocity distribution.

147. Shock and Eddy Losses.—In a *distorted model* constructed with a single scale ratio for horizontal dimensions and another for vertical dimensions, horizontal angles are the same in the model and in nature. The formation of eddies with vertical axes may be expected to be similar, if Eq. (10.8) is satisfied and the velocities in the eddies are sufficient to result in turbulent flow. However, vertical angles are increased and the corresponding loss of energy in eddy formation is disproportionately greater. In Chap. VII the loss in diverging sections was found to be proportional to V^2, but the coefficient of loss varied with the angle and showed a very sharp minimum point. If m such losses occur in a certain length, the total loss may be written as

$$h_s = mKV^2$$
$$b_z = b_K b_V^2$$

Here, $b_m = 1$ because the same number of expansions and contractions will occur in corresponding lengths. To satisfy Eq. (10.7), $b_K = 1$ but in nearly all distorted models $b_K > 1$, and the loss is greater than it should be for similarity. Since these points of shock loss are not uniformly distributed over the whole bed, local adjustments of roughness to secure a certain water-surface curve as measured at the banks will not compensate exactly for the excessive shock loss. A simple explanation of this discrepancy is that eddies with a horizontal axis are probably circular in both model and prototype instead of being elliptical in a distorted model as required by the scale ratios.

148. Critical Depth.—In Chap. IX, it was shown that the critical depth is critical from almost every viewpoint. At this depth the water velocity equals the velocity of a small wave, the momentum and specific energy show minimum values, the discharge is a maximum, and the slope of the water surface becomes very great. Considering these circumstances, it is evidently important to investigate whether the distortion of a model has reduced the depth below the critical depth. In nature, the

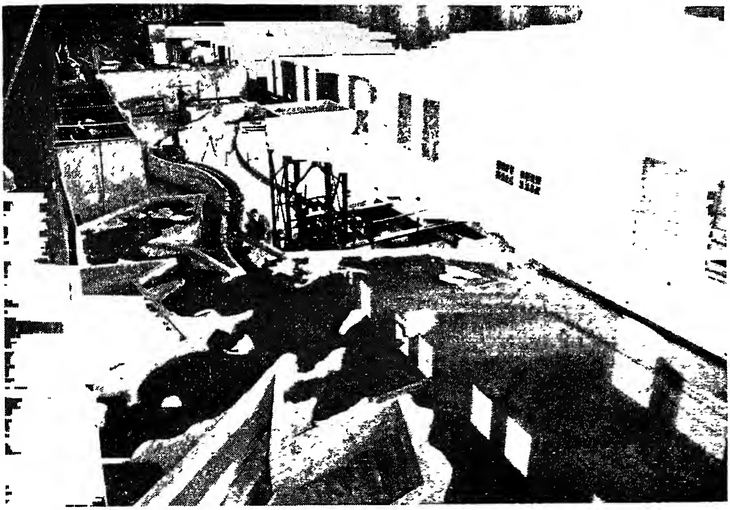

FIG. 160.—Model of the Cape Cod Canal, built under the direction of Prof. K. C. Reynolds, Department of Civil and Sanitary Engineering, M. I. T., in cooperation with the Corps of Engineers, U. S. A. Horizontal scale 1:600. Vertical scale 1:60.

velocity is less than the critical velocity in nearly all problems suitable for solution by distorted models and therefore the velocity in the model must also be less than the critical value. The velocity at the critical depth is

$$c = \sqrt{gD_m}, \qquad b_c = b_z^{\frac{1}{2}}$$

It has been shown previously that the criterion for similarity of distorted models is that $b_V = b_z^{\frac{1}{2}}$ and it follows that the ratio of actual to critical velocity is identical in model and prototype since they are both reduced by the same scale factor.

The conclusion drawn above is contrary to the usual statements regarding distorted models to the effect that by excessive distortion the velocity in the model might exceed the wave velocity. Following Winkel, the reasoning behind this conclusion is as follows: The mean velocity given by the friction equation is

$$V = K\sqrt{RS}$$

The velocity of a wave is $\sqrt{gD}$. Assuming $R = D$, the requirement that the velocity be less than the critical value may be expressed as

$$V = K\sqrt{DS} < \sqrt{gD}$$
$$S < \frac{g}{K^2}$$

However, this reasoning neglects the fact that K is reduced by increasing the roughness to a point such that $b_V{}^2 = b_z$. Expressing K by means of Manning's equation,

$$K = \frac{1.49}{n}R^{\frac{1}{6}}$$

From Eq. (10.8),

$$b_n = b_z{}^{\frac{2}{3}}b_x{}^{-\frac{1}{2}}$$

and

$$b_K = b_x{}^{\frac{1}{2}}b_z{}^{-\frac{1}{2}}$$

Under this condition the ratio of water velocity to wave velocity is the same in the model as in the prototype, as can be shown by writing the ratio:

$$\frac{\dfrac{V}{c}}{\dfrac{v}{c}} = \frac{\dfrac{K\sqrt{DS}}{\sqrt{gD}}}{\dfrac{k\sqrt{dS}}{\sqrt{gd}}} = b_K b_S{}^{\frac{1}{2}} = 1$$

since

$$b_S = b_z b_x{}^{-1}$$

and

$$b_K = b_x{}^{\frac{1}{2}}b_t{}^{-\frac{1}{2}}$$

The value of n and K cannot be independently controlled in movable-bed models and because of this fact the ratio of water velocity to wave velocity is changed and the critical depth may result if the model discharge is distorted from $b_Q = b_x b_z^{3/2}$ to give proper slope. The other method of correcting for friction is to set off the model discharge by $b_Q = b_x b_z^{3/2}$ and adjust the backwater to the proper stage, a procedure which results in a slope less than given by $b_z b_x^{-1}$. The latter method preserves the ratio

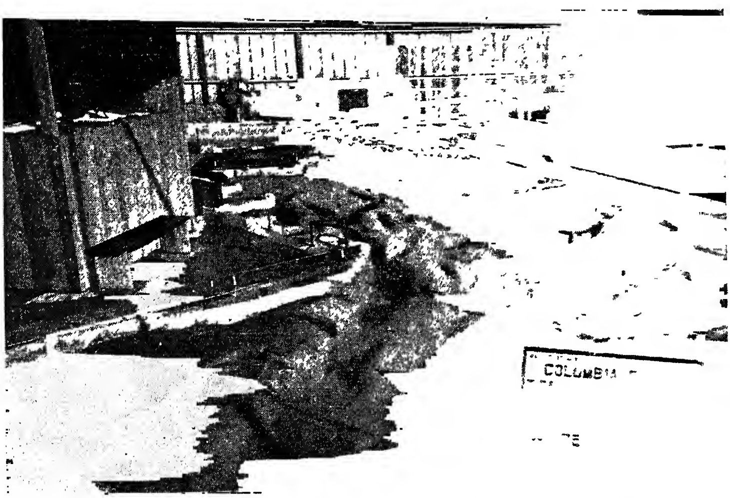

FIG. 161.—Model of Columbia River Estuary. Horizontal scale 1:3600. Vertical scale 1:128. (*Courtesy U. S. Tidal Model Laboratory.*)

of average water velocity at one point only, but the distortion of this ratio at other points is not ordinarily sufficient to produce the critical depth.

149. Models with Movable Bed.—Models have been extensively used to study problems of river regulation, particularly those involving the scour or fill, as well as the currents resulting from dikes, dredging, cutoffs, or other improvements. Such models are frequently formed in mobile materials with the intention of securing quantitative data on the changes produced.

Unfortunately, present empirical formulas for the movement of granular materials by flowing water are based largely on laboratory experiments and very few data obtained under natural conditions are available for checking them. In the

absence of confirming field data covering a range of materials, depths, and velocities, these equations cannot be accepted as a satisfactory basis for the formulation of model laws and for this reason movable-bed models will be treated only in general terms.

Granular materials such as sand behave differently from liquids in that they exhibit a yield point when subjected to tangential stress. Below this yield point, no movement occurs. This fact is of importance in the design of movable-bed models because too great a reduction in the size of the model may result in tractive forces which are smaller than the critical tractive force necessary to start movement. The equation for the tractive force is

$$\tau_0 = wRS \tag{10.9}$$

From Manning's equation,

$$S = \frac{n^2V^2}{(1.49)^2R^{\frac{4}{3}}}$$

Substituting in Eq. (10.9),

$$\tau_0 = \frac{wn^2V^2}{(1.49)^2R^{\frac{1}{3}}}$$

$$b_{\tau 0} = \frac{b_n{}^2b_V{}^2}{b_R{}^{\frac{1}{3}}} \cong b_n{}^2b_z{}^{\frac{2}{3}}$$

The scale of tractive force depends principally on the roughness and the vertical scale ratio and on the horizontal scale ratio only to the extent that this ratio affects the ratio b_R. This fact accounts in large part for the distortion of movable-bed models because by this means the tractive forces in the model are greater than they would be if the vertical scale were the same as the horizontal.

Many attempts have been made to express the critical tractive force in terms of the specific gravity, shape, and size of the material, but none have been entirely successful. There is difficulty even in formulating a definition of the critical condition because the whole sand mixture does not start to move simultaneously. Scattering of the experimental data is so considerable that a mean line drawn through the points has little significance, because a particular sand chosen for model use might well have a critical tractive force considerably above this averaged value.

An equation given by Krey for the tractive force necessary to start movement of quartz sand in water is

$$\tau_c = 0.6d$$

where τ_c = the critical tractive force, pounds per square foot.
d = the median diameter, inches (50 per cent on cumulative percentage diagram).

It is not sufficient for similarity of scour that the average tractive force in the model exceed the critical value. In the first place the percentage of the total area over which sand movement occurs should be the same as in nature and secondly, the depth

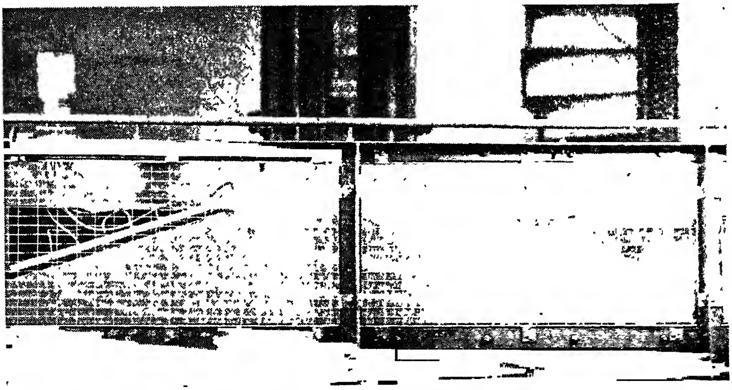

FIG. 162.—Model of a permeable earth dike. Dyes indicate the paths of flow. (*Courtesy Lorenz G. Straub.*)

of scour probably depends on the ratio of the actual local tractive force to the critical value. Both of these considerations suggest that the proper criterion of similarity is that the ratio of the actual tractive force to the critical value be the same in model and prototype. This is a very stringent condition and one seldom satisfied in river models. If the same material were used, it would be necessary to increase the vertical scale and the slope so as to produce the same tractive force in model and nature. To indicate the approximate reduction in size of material as a function of the vertical scale to meet this requirement,

$$b_{\tau c} = b_d = b_{\tau 0} = b_n^2 b_z^{\frac{2}{3}}$$

Strickler's equation for n is $n \propto d^{\frac{1}{6}}$ and on this basis $b_n^2 = b_d^{\frac{1}{3}}$ and $b_d = b_z$. These results are rough approximations and are

included only to indicate the method of application of information on bed transportation to model studies and to emphasize the difficulty of satisfying all of the criteria simultaneously. For example, in a distorted, movable-bed model, $b_V{}^2 < b_z$ because the friction coefficient does not increase as it should for similarity and the tractive forces are proportionately increased, which is advantageous as regards bed movement. However, the condition for similarity of distorted models is that $b_V{}^2 = b_z$ and deviations from this relationship must cause some dissimilarity in the paths of flow.

An examination of data comparing distorted, movable-bed models with their prototypes leads to the conclusion that, at present, such models can yield but qualitative information and that only when designed and operated with proper regard for the various criteria mentioned.

GENERAL CONSIDERATIONS

150. Reliability of Models.—The attitude of American engineers towards the use of hydraulic models has changed markedly in the last ten years and confidence in them has risen to a point which is not entirely justified. When properly designed and operated, they are valuable adjuncts to analysis and judgment, but they are not infallible. It is not sufficient to run water through something which looks like the prototype to have a working model; and yet this is precisely what has been done in altogether too many instances. For example, the backwater curve fixed by downstream controls is of basic importance in almost every model experiment and yet extensive investigations have been made without precise information on this point. Another frequent error is the failure to make provisions for obtaining the proper distribution of velocities at inlet and outlet. Other reasons for disagreement between models and their prototypes are mentioned in the preceding notes.

In spite of the large expenditures for model testing and for works based on such tests, there exists no reliable quantitative set of criteria establishing the minimum satisfactory size of models of different types. The cost of a model test increases rapidly with size, and the aim of model design should be to select the smallest size which will yield reliable information; but the

data necessary for such economic planning of model experiments are not available and in the final analysis the design is largely an intuitive process aided by general hydraulic knowledge and previous experience with similar models shown to be reliable by direct quantitative comparison with the prototype. Unfortunately, the number of such comparisons is limited and there is almost no information on models which were definitely unsatisfactory. Most of these comparisons, however, have shown reasonably good quantitative agreement of the features compared. The reliability improves as the model size is increased and the distortion ratio decreased, and the safe procedure is to make the model as large as possible.

The criteria mentioned above are those which are needed to design a model. If the model represents a proposed structure which has no relationship to existing conditions, its reliability cannot be established by comparison. However, a relatively large number of models are concerned with modifications of existing conditions and under such circumstances it is possible to compare the model with nature before modifications are made. If the model agrees with nature in the original condition, it is reasonable to expect that it will agree afterward and that the percentage differences will be approximately the same for the two conditions. Even this procedure is not entirely satisfactory because the proposed changes may be of such a nature as to materially alter the relative importance of friction and of shock or eddy losses. For example, if a movable-bed model is to be used to study the effect of a set of dikes in a river, one might first determine whether with proper hydraulic conditions the model will form a bed corresponding to that existing in nature. If it does, then one can expect that the model will also reproduce the effect of the dikes and it might even show better agreement than in the original condition because the dikes would tend to increase the importance of shock and eddy losses.

In spite of the many difficulties and uncertainties involved, the use of properly designed models is believed to be superior to any other method of attack short of full-scale experimentation in the field. Experience in similar situations is, of course, the approximate equivalent of field experimentation. Theoretical analysis can be applied only to relatively simple problems and there are always questionable assumptions to be made. The

accuracy of unaided judgment is probably less than that of even the worst models.

151. Desirable Experiments.—The preceding sections have indicated certain deficiencies in the model theory and in the hope of stimulating work in this field, the following list of experiments has been prepared.

1. Quantitative comparison of series of geometrically similar models of different types, such as rivers, tidal canals, chutes, dams, and so forth.
2. Experiments in small channels to determine friction losses, critical Reynolds number, and lower limit of fully developed turbulence.
3. Determination of the conditions under which surface tension becomes important.
4. The effect of distortion on velocity distribution and on the reliability of models of different types. The distortion should be carried to extreme conditions.
5. Extension of bed-load experiments to larger channels to determine the general laws involved.
6. Comparison of the scour and fill in distorted and undistorted models of different sizes.
7. Repetition of runs under apparently identical hydraulic conditions to determine reproducibility of results of movable-bed models.
8. Comparison of models and prototypes. Verification tests should be reported when made, and structures built as a result of model tests should be tested whenever possible. Failures as well as good agreement should be reported in full in order that the model laws may be firmly established on an experimental as well as a theoretical basis.

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Problems

1. The discharge over a dam is given by the formula $Q = CLH^{\frac{3}{2}}$ where C is the discharge coefficient and L the length of crest. The discharge over a model which is reduced to 1: 30 in all its dimensions is 1.0 cu. ft. per sec. What is the corresponding discharge of the prototype?

2. A distorted hydraulic model is to be constructed with a horizontal scale of 1:1240 and a vertical scale of 1:100. Determine the ratio of discharges.

3. Prepare a critical summary of the report on one of the model experiments listed below:

(*a*) Baffle-Pier Experiments on Models of Pit River Dams, I. C. Steele and R. A. Monroe, *Trans. Am. Soc. Civil Eng.*, vol. 93, p. 451 1929.

(*b*) Experiments on Discharge over Spillways and Models, Keokuk Dam, F. A. Nagler and Albion Davis, *Trans. Am. Soc. Civil Eng.*, vol. 94, p. 777, 1930.

(*c*) Model Law for Motion of Salt Water through Fresh, M. P. O'Brien and John Cherno, *Trans. Am. Soc. Civil Eng.*, vol. 99, p. 576, 1934.

(*d*) Tests of Broad-crested Weirs, J. G. Woodburn, *Trans. Am. Soc. Civil Eng.*, vol. 96, p. 387, 1932.

(*e*) Sand Mixtures and Sand Movement in Fluvial Models, Hans Kramer, *Trans. Am. Soc. Civil Eng.*, vol. 100, p. 798, 1935.

4. Assuming that the only restriction on model design is a maximum flow of 3 cu. ft. per sec., proportion a model for one of the structures listed below. Check against criteria. (*a*) Boulder Dam spillways, (*b*) Norris Dam spillways, (*c*) Desilting structures at the headworks of the All-American Canal, (*d*) Bonnet Carre spillway on the Mississippi River near New Orleans, (*e*) Bonneville Dam spillways. Descriptions of these structures may be found in recent articles and papers in *Eng. News-Record, Civil Eng.*, and *Proc., Am. Soc. Civil Eng.*

APPENDIX I

PHYSICAL PROPERTIES OF FLUIDS

Quantitative measurements of the physical properties of some fluids are given here to facilitate the solution of problems. For more detailed information, consult the following reference works:

International Critical Tables.

Smithsonian Physical Tables.

Handbook of Chemistry and Physics.

Hydraulic Tables, University of California Syllabus Series, No. 246.

1. The *compressibility* of liquids is usually expressed by means of the bulk modulus, which is defined mathematically as:

$$K = \frac{-\Delta p}{\dfrac{\Delta V}{V}}$$

Here Δp is the small change in pressure necessary to change a volume V by an amount ΔV. The bulk moduli of water and mercury appear in Table 4.

TABLE 4.—BULK MODULUS

Liquid	Pressure, atm.	Temperature, deg. Cent.	Bulk modulus, lb./sq. in.	Source
Water	1–25	0	270,000	Amagat
Water	25–50	0	285,000	Amagat
Water	100–200	0	300,000	Amagat
Water	500–1000	0	355,000	Amagat
Water	1000–1500	0	410,000	Amagat
Water	1500–2000	0	455,000	Amagat
Water	2000–2500	0	505,000	Amagat
Water	2500–3000	0	565,000	Amagat
Water	1–25	10	294,000	Amagat
Water	1–25	20	300,000	Amagat
Water	25–50	10	298,000	Amagat
Mercury	1–1000	0	$\begin{pmatrix} 3.75 \times 10^6 \\ 3.8 \times 10^6 \end{pmatrix}$	Bridgman
Mercury	1000–2000	0	3.92×10^6	Bridgman

Since gases follow the perfect gas law with sufficient accuracy for most engineering computations, their compressibility can be expressed in terms of this law in place of the bulk modulus. The equation is

$$pv = RT$$

where p = the pressure, pounds per square foot.
v = the volume of a unit weight of gas, cubic feet per pound.
R = the gas constant, foot-pounds per pound degree Rankine.
T = the absolute temperature, degrees Rankine.

TABLE 5.—COMPRESSIBILITY OF SOME GASES

Gas	c_p*	c_v*	n	R†	v,‡ cu. ft./lb.	w,‡ lb./cu. ft.
Air..........................	0 240	0.1715	1.40	53.3	12.39	0.0807
Helium........................	1.251	0.754	1.659	386.6	90.0	0.0111
Hydrogen......................	3.140	2.160	1.46	762.0	177.1	0.00565
Oxygen........................	0.2175	0.1553	1.40	48.2	11.22	0.0891
Carbon dioxide................	0 2025	0.1575	1.29	35 1	8 15	0.1227

* See p. 342 for definitions of c_p and c_v.
† The numerical value of the gas constant R is approximately 1544 divided by the molecular weight.
‡ At 32 deg. Fahr. and 14.7 lb. per sq. in.

2. *Density* is defined as the mass per unit volume. It may vary from point to point in a fluid but at any point it has a definite value and the variation in this value is assumed to be continuous. Remembering that fluids are composed of molecules, it is evident that we are not precisely correct in speaking of the density at a point since this value might be zero or extremely great, depending on whether the region surrounding the point is occupied by a molecule. However, the great number of molecules present in even the smallest volumes used in engineering permits the assignment of the average density in a region to particular points.

The density of liquids depends upon the pressure, temperature, and the percentage of foreign substances present. Values for pure water at atmospheric pressure and for a few other common liquids are given in Table 6.

Sea water has an average specific gravity of about 1.025 due to the dissolved salts, giving it a unit weight of about 64.0 lb. per cu. ft. It should be noted that density is expressed as units of mass per cubic foot, and that the unit weight is obtained by multiplying the density by the gravitational force per unit mass. Specific gravity is the ratio of the density of the material in question to that of distilled water at a standard temperature and pressure.

The density of a gas in the ordinary range of temperatures and pressures is obtained from the perfect gas law. Since v is the volume of a unit weight of the gas, the density is

$$\rho = \frac{p}{gRT}$$

The unit weight is

$$w = \frac{p}{RT} = \rho g$$

For air, the National Advisory Committee for Aeronautics defines "standard atmosphere" as fulfilling the following conditions:

Pressure:	29.92 in. of mercury.
Temperature:	59 deg. Fahr.
Absolute temperature:	518.4 deg. Rankine.
Gravity:	32.174 ft. per sec.^{2}
Unit weight:	0.0765 lb. per cu. ft.
Density:	0.002378 slugs per cu. ft.
Temperature gradient:	0.003566 deg. Fahr. per ft.
Gas constant:	53.33 ft. per deg. Rankine.

3. The *viscosity* of a fluid is defined as its resistance to internal forces. If the space between the two flat plates B and C is filled with a fluid of viscosity μ, the force necessary to move the plate B parallel to the plate C at a constant velocity V is (Fig. 163)

$$F = \mu A \frac{V}{z}$$

where F is the force and A is the area of the plate B.

It should be noted that the force is directly proportional to the coefficient of viscosity, the area, and the velocity, and inversely proportional to the distance between the plates. In special cases, the velocity distribution is linear, provided that

V does not exceed a certain critical value, but in general, thin layers of the liquid must be considered. The equation then becomes

$$F = \mu A \frac{dV}{}$$

In many problems of fluid mechanics, the ratio of the viscous forces to the inertia forces is of primary importance and for these cases the kinematic viscosity is used in place of the absolute

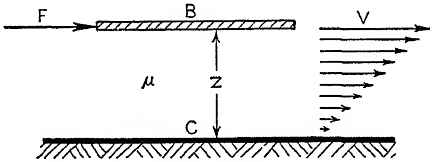

FIG. 163.

viscosity. Representing the kinematic viscosity by ν, its value is given by the expression

$$\nu = \frac{\mu}{\rho}$$

The kinematic and the absolute viscosities of liquids vary with the temperature but not appreciably with the pressure. The values for a few common liquids are given in Table 6.

The absolute viscosity of gases is found to vary with temperature but not with pressure, but for any temperature the kinematic viscosity varies inversely as the pressure.

4. *Cohesion, adhesion,* and *surface tension* are related to the molecular properties of fluids. Cohesion is the tendency of the particles of a solid or fluid to cling together while adhesion is their tendency to attach themselves to other objects. Familiar examples illustrating the relative forces exerted by cohesion and adhesion are the wetting of glass by water, in which the adhesive force between glass and water is great, and the nonwetting of glass by mercury in which the adhesive force between glass and mercury is small, compared with the cohesive forces of the liquid. Surface tension is a result of the cohesion of the particles and exhibits itself as a tendency for the surface of contact with other liquids or solids to become as small as possible. The value of

TABLE 6.—DENSITY AND VISCOSITY OF SOME LIQUIDS

Liquid	Temperature, deg. Fahr.	Density, slugs/ft.^{3}	Unit wt., lb./ft.^{3}	Absolute viscosity, lb. sec./ft.^{2}	Kinematic viscosity, ft.^{2}/sec.
Water	32	1.940	62.42	3.746×10^{-5}	1.931×10^{-5}
	41	1.940	62.42	3.180×10^{-5}	1.639×10^{-5}
	50	1.939	62.41	2.736×10^{-5}	1.411×10^{-5}
	59	1.937	62.36	2.383×10^{-5}	1.230×10^{-5}
	68	1.934	62.31	2.098×10^{-5}	1.085×10^{-5}
	77	1.930	62.24	1.867×10^{-5}	0.967×10^{-5}
Castor oil	41	1.883	60.60	0.0775	0.0416
	50	1.876	60.40	0.0503	0.0268
	59	1.870	60.20	0.0316	0.0169
	68	1.863	60.00	0.0205	0.0110
Mercury	−4	26.5	853	....	1.463×10^{-6}
	32	......	......	3.50×10^{-5}	1.323×10^{-6}
	68	......	......	3.30×10^{-5}	1.237×10^{-6}
	122	......	......	2.98×10^{-5}	1.122×10^{-6}
	212	25.9	834	2.53×10^{-5}	0.982×10^{-6}
	392	......	......	2.12×10^{-5}	0.827×10^{-6}
	572	24.8	799	1.94×10^{-5}	0.775×10^{-6}
Zerolene, No. 3	40	1.823	58.76		
	60	1.809	58.28		
	80	1.795	57.81	2.54×10^{-3}	1.41×10^{-3}
	100	1.780	57.36	1.33×10^{-3}	0.75×10^{-3}
	120	1.766	56.92	0.72×10^{-3}	0.41×10^{-3}
	150	1.745	56.26	0.30×10^{-3}	0.17×10^{-3}
	200	1.710	55.15	0.15×10^{-3}	0.09×10^{-3}

TABLE 7.—ABSOLUTE VISCOSITIES OF SOME GASES

Multiply tabular values by 10^{-9} pound-seconds per square foot

Temperature deg. Fahr.	Air	Hydrogen	Nitrogen	Carbon dioxide
32	356	179	346	286
50	366	184	357	296
68	376	188	368	306
86	386	193	377	316
104	395	197	386	326
140	413	206	405	346
176	432	215	424	365
212	440	224	443	383

the surface tension depends upon the relation between the cohesion of the fluid and its tendency to diffuse through the fluid with which it is in contact. That is, it is a result of the forces exerted on the surface molecules by those in the interior of the fluid and the forces exerted by the molecules of the substance with which it is in contact. The force of surface tension is always tangential to the surface of separation and has the same magnitude perpendicular to any imaginary line in the surface.

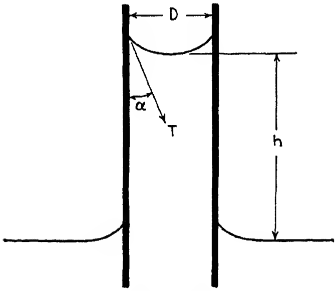

FIG. 164.

Surface tension is responsible for the phenomenon known as *capillarity*. Referring to Fig. 164 the height to which a liquid will rise in a circular tube is shown by Gibson[1] to be very nearly

$$h = \frac{4T \cos \alpha}{wd}$$

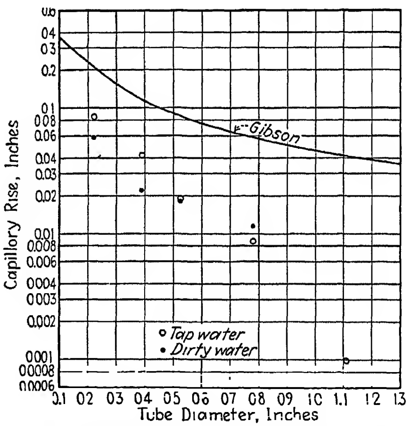

FIG. 165.—Capillary rise in glass tube. Experiment of Folsom compared with Gibson's equation.

Here T is the tension of the liquid-air interface and α is the angle made by this surface with the wall of the tube. The height h is a correction which must be made to the readings of manom-

[1] Gibson, A, H.: "Hydraulics and Its Applications," 4th ed. p. 8, D. Van Nostrand Company, Inc., New York.

eters. In practice, it is not safe to rely on a calculated correction but to make the manometer tube large enough so that the correction is negligible. The value of surface tension changes considerably due to impurities in the liquid, and foreign matter, such as grease or dust on the walls of the tube. Figure 165 shows the results of experiments made by Folsom[1] compared with Gibson's equation.

TABLE 8.—VALUES OF SURFACE TENSION OF FLUIDS IN CONTACT WITH AIR

Fluid	Temperature, deg. Fahr.	Specific gravity	Surface tension	
			dynes/cm.	lb./ft.
Water	32	.....	75.6	0.00518
	68	.....	72.8	0.00498
	104	.....	70.0	0.00480
	140	.....	67.1	0.00460
	176	.....	64.3	0.00441
	212	.....	61.5	0.00421
Sodium chloride in water	68	1 193	85.8	0.00588
	68	1.107	80.5	0.00552
Ethyl alcohol	32	.....	23.5	0.00161
	68	.....	21.7	0.00149
	104	.....	20.0	0.00137
	140	.....	18.2	0.00125
Mercury	68	.....	465	0.0318

5. The ability of water to take air, carbon dioxide, sodium chloride, and many other substances into solution is of immense biological and economic importance. In experiments on hydraulic machinery, siphon spillways, and other apparatus, the presence of dissolved gases which are liberated at points of low pressure imposes very distinct limitations on the manner of operation. The solubility of gases in water varies with the temperature and pressure and with the kind of gas.

6. The *specific heat* of a substance is the amount of heat necessary to raise the temperature of a unit mass through 1 deg. of

[1] Folsom, R. G.: Manometer Errors Due to Capillarity, *Instruments*, February, 1936, pp. 36–37.

TABLE 9.—SOLUBILITY OF AIR IN PURE WATER

Temperature, deg. Fahr.	32	50	86	158	212
Volume of air at 0 deg. Cent. and 760 mm. per unit volume of water at 760 mm.........	0.029	0.023	0.016	0.012	0.011

temperature. The normal calorie, which is the usual unit of heat, is the amount of heat required to raise 1 gram of pure water from 15 to 16 deg. Cent. The specific heat of liquids does not vary greatly with changes in pressure and temperature.

TABLE 10.—SPECIFIC HEAT OF WATER IN NORMAL CALORIES (15 DEG. CENT.)

Temperature, deg. Cent.	Specific heat
0	1.00874
10	1.00184
15	1.00000
25	0.99765
40	0.99761
60	0.99934
80	1.00239
100	1.00645

Since gases change either volume or pressure or both with changes in temperature, the heat per unit weight necessary to accomplish a given temperature change depends upon the amount of external work done during the heating. Two specific heats are used, one known as the specific heat at constant volume, c_v, and the second as the specific heat at constant pressure, c_p. These specific heats of gases are not constant but are nearly enough so for most engineering computations. Specific heats of some common gases are listed under Compressibility (Table 5).

7. The *boiling point* of liquids is that temperature above which the cohesive forces are no longer able to hold the molecules tightly compacted and they escape to form a vapor or a gas. Under atmospheric pressure, the boiling point of water is 100 deg. Cent. or 212 deg. Fahr. and increases with increase of pressure. However, below the boiling point, molecules escape from the liquid in spite of the surface tension and permeate the volume

exposed to the liquid. This process continues until the partial pressure due to the vapor reaches a certain maximum value dependent upon the temperature. This tendency for water and other liquids to give off a vapor at temperatures below the boiling point is important in determining the suction lift of pumps, the elevation of pipelines and siphon spillways, and in the water losses from exposed tanks, reservoirs, and ground surfaces.

Table 11.—Vapor Pressure of Water

Temperature, deg. Fahr.	Vapor pressure, ft. of water
32	0.204
50	0.407
68	0.774
86	1.405
104	2.45
122	4.10
140	6.64
168	10.40
186	15.84
204	23.4
212	33.9

APPENDIX II

HYDRAULIC TABLES

TABLE 12.—DIMENSIONS OF WROUGHT-IRON AND STEEL STEAM, GAS, AND WATER PIPE

Diameter			Transverse areas	
Nominal internal, in.	Approximate internal, in.	Actual external, in.	External, sq. in.	Internal, sq. in.
$\frac{1}{8}$	0.269	0.405	0.129	0.057
$\frac{1}{4}$	0.364	0.540	0.229	0.104
$\frac{3}{8}$	0.493	0.675	0.358	0.191
$\frac{1}{2}$	0.622	0.840	0.554	0.304
$\frac{3}{4}$	0.824	1.050	0.866	0.533
1	1.049	1.315	1.358	0.864
$1\frac{1}{4}$	1.380	1.660	2.164	1.495
$1\frac{1}{2}$	1.610	1.900	2.835	2.036
2	2.067	2.375	4.430	3.355
$2\frac{1}{2}$	2.469	2.875	6.492	4.788
3	3.068	3.500	9.621	7.393
$3\frac{1}{2}$	3.548	4.000	12.566	9.886
4	4.026	4.500	15.904	12.730
$4\frac{1}{2}$	4.506	5.000	19.635	15.947
5	5.047	5.563	24.306	20.006
6	6.065	6.625	34.472	28.891
7	7.023	7.625	45.664	38.738
8	7.981	8.625	58.426	50.027
9	8.941	9.625	72.760	62.786
10	10.020	10.750	90.763	78.855
11	11.000	11.750	108.434	95.033
12	12.000	12.750	127.676	113.097

TABLE 13.—VALUES OF f FOR USE IN THE WEISBACH EQUATION $h_L = f\frac{L}{D}\frac{V^2}{2g}$ AS DETERMINED BY FANNING FOR STRAIGHT SMOOTH PIPES[1]

Diam. of pipe, in.	Mean velocity V, ft./sec. 0.5	1.0	2.0	3.0	4.0	5.0	10.0	15.0	20.0
0.5	0.0418	0.0381	.0340	.0317	0.0300	0.0287	0.0250	0.0237	0.0231
0.75	0.0405	0.0366	.0329	.0308	0.0292	0.0280	0.0247	0.0235	0.0229
1.	0.0398	0.0353	.0317	0.0300	0.0285	0.0274	0.0245	0234	0.0228
1.5	0.0384	0.0343	.0310	0.0292	0.0278	0.0268	0.0241	0.0231	0.0226
2.	0.0364	0.0330	0.0301	0.0284	0.0272	0.0263	0.0237	0.0228	0.0223
3.	0.0354	0.0317	0.0288	0.0273	0.0263	0.0254	0.0232	0.0224	0.0219
4.	0.0340	0.0306	0.0279	0.026	0.0255	0.0247	0.0226	0.0219	0.0214
5.	0.0328	0.0297	0.0271	0.0258	0.0249	0.0241	0.0222	0.0215	0.0211
	0.0317	0.0289	0.0264	0.025	0.0243	0.0236	0.0219	0.0212	0.0208
	0.0296	0.0275	0.0253	0.024	0.0234	0.0227	0.0212	0.0207	0.0202
10.	0.0283	0.0262	0.024	0.0232	0.0225	0.0220	0.0206	0.0201	0.0197
12.	0.0268	0.0250	0.0233	0.0224	0.0218	0.0213	0.0201	0.0196	0.0192
14.	0.0256	0.0241	0.022	0.0217	0.0211	0.0207	0.0196	0.0192	0.0188
16.	0.0244	0.023	0.0218	0.0210	0.0205	0.0201	0.0192	0.0188	0.0184
18.	0.0236	0.0224	0.0211	0.0204	0.0199	0.0196	0.0188	0.0183	0.0181
20.	0.0229	0.0216	0.0204	0.0198	0.0194	0.0191	0.0184	0.0180	0.0177
24.	0.021	0.020	0.0193	0.0187	0.0184	0.0182	0.0176	0.0173	0.0170
30.	0.0194	0.0186	0.0179	0.0175	0.0173	0.0171	0.0166	0.0163	0.0161
36.	0.0177	0.0171	0.0166	0.016	0.0162	0.0161	0.0156	0.0154	0.0152
42.	0.0164	0.0160	0.0156	0.0154	0.0153	0.0152	0.0148	0.0146	0.0145
48.	0.0153	0.0150	0.0147	0.0146	0.0145	0.0144	0.0141	0.0139	0.0138
54.	0.0144	0.0142	0.014	0.013	0.0137	0.0137	0134	0.0133	0.0132
60.	0.013	0.013	0.0133	0.0132	0.0131	0.0131	0128	0.0127	0.0125
72.	0.0126	0.0124	0.0122	0.012	0.0120	0.0119	0.0117	0.0117	0.0117
84.	0.011	0.0115	0.011	0.0112	0.0112	0.0111	0.0109	0.0109	0.0108

[1] From "Handbook of Hydraulics" by King, 2d ed.

TABLE 14.—APPROXIMATE VALUES OF THE PIPE COEFFICIENT f FOR CAST- AND WROUGHT-IRON PIPES IN FAIR INTERIOR CONDITION[1]

Velocity, ft./sec.	Actual diameter, in.																	
	3	4	5	6	8	10	12	14	16	18	20	24	30	36	42	48	54	60
0.5	.0384	.0356	.0335	.0322	.0299	.0284	.0271	.0261	.0252	.0245	.0238	.0228	.0215	.0206	.0198	.0192	.0186	.0180
1.0	.0347	.0322	.0304	.0291	.0271	.0257	.0245	.0237	.0228	.0221	.0215	.0207	.0195	.0187	.0179	.0173	.0168	.0164
1.5	.0328	.0305	.0286	.0275	.0256	.0243	.0231	.0224	.0215	.0209	.0204	.0195	.0184	.0176	.0169	.0164	.0159	.0154
2.0	.0315	.0292	.0275	.0264	.0246	.0234	.0222	.0215	.0207	.0200	.0196	.0187	.0177	.0169	.0162	.0157	.0153	.0148
2.5	.0305	.0284	.0267	.0256	.0238	.0226	.0215	.0208	.0200	.0195	.0190	.0182	.0171	.0164	.0157	.0152	.0148	.0143
3.0	.0298	.0277	.0260	.0250	.0233	.0221	.0210	.0202	.0196	.0190	.0185	.0177	.0167	.0160	.0153	.0148	.0144	.0140
3.5	.0292	.0271	.0255	.0245	.0228	.0216	.0205	.0198	.0191	.0186	.0181	.0174	.0163	.0157	.0150	.0145	.0141	.0137
4.0	.0286	.0266	.0250	.0240	.0223	.0212	.0201	.0195	.0188	.0182	.0178	.0170	.0160	.0154	.0147	.0143	.0139	.0134
5.0	.0277	.0258	.0242	.0233	.0216	.0205	.0196	.0189	.0182	.0177	.0172	.0165	.0156	.0149	.0143	.0138	.0134	.0130
6.0	.0270	.0252	.0236	.0227	.0211	.0200	.0191	.0184	.0178	.0172	.0168	.0161	.0152	.0145	.0139	.0135	.0131	.0127
7.0	.0265	.0246	.0231	.0222	.0207	.0196	.0187	.0180	.0174	.0169	.0164	.0158	.0148	.0142	.0136	.0132	.0128	.0124
8.0	.0260	.0242	.0227	.0218	.0203	.0193	.0183	.0177	.0170	.0166	.0161	.0155	.0146	.0140	.0134	.0129	.0126	.0122
10.0	.0252	.0234	.0220	.0212	.0197	.0187	.0178	.0172	.0165	.0160	.0156	.0150	.0142	.0135	.0129	.0126	.0122	.0118
12.0	.0245	.0228	.0214	.0206	.0192	.0182	.0173	.0167	.0161	.0157	.0152	.0146	.0138	.0132	.0127	.0122	.0119	.0115
16.0	.0238	.0220	.0206	.0198	.0185	.0177	.0166	.0160	.0155	.0150	.0146	.0140	.0132	.0127	.0122	.0118	.0114	.0111
20.0	.0229	.0212	.0199	.0192	.0178	.0170	.0161	.0155	.0150	.0146	.0142	.0136	.0128	.0123	.0118	.0114	.0110	.0107

[1] From "Hydraulic Tables," University of California Syllabus Series No. 246.

TABLE 15.—APPROXIMATE VALUES OF THE PIPE COEFFICIENT f FOR RIVETED PIPES, AND FOR CAST- AND WROUGHT-IRON PIPES AFTER BEING IN SERVICE ABOUT 10 YEARS[1]

Velocity, ft./sec.	Actual diameter, in.																	
	3	4	5	6	8	10	12	14	16	18	20	24	30	36	42	48	54	60
0.5	.0473	.0438	.0416	.0396	.0369	.0349	.0333	.0321	.0310	.0301	.0294	.0282	.0266	.0253	.0245	.0236	.0229	.0224
1.0	.0457	.0423	.0401	.0383	.0357	.0337	.0322	.0310	.0299	.0291	.0283	.0272	.0257	.0245	.0236	.0228	.0221	.0216
1.5	.0448	.0415	.0394	.0375	.0350	.0330	.0315	.0304	.0293	.0285	.0277	.0266	.0252	.0240	.0231	.0223	.0217	.0211
2.0	.0441	.0409	.0388	.0370	.0345	.0326	.0311	.0299	.0289	.0281	.0274	.0263	.0248	.0236	.0228	.0220	.0214	.0208
2.5	.0437	.0403	.0384	.0366	.0341	.0322	.0307	.0296	.0286	.0278	.0270	.0260	.0245	.0234	.0225	.0217	.0211	.0206
3.0	.0432	.0400	.0380	.0363	.0338	.0319	.0305	.0293	.0283	.0276	.0268	.0257	.0243	.0232	.0223	.0215	.0210	.0204
3.5	.0429	.0398	.0378	.0360	.0335	.0317	.0302	.0291	.0281	.0274	.0266	.0255	.0242	.0230	.0222	.0214	.0208	.0202
4.0	.0427	.0395	.0375	.0357	.0333	.0315	.0300	.0289	.0279	.0272	.0264	.0253	.0240	.0228	.0220	.0212	.0207	.0201
5.0	.0422	.0392	.0371	.0353	.0330	.0312	.0297	.0286	.0277	.0269	.0261	.0251	.0237	.0226	.0218	.0210	.0204	.0199
6.0	.0419	.0388	.0368	.0351	.0327	.0309	.0295	.0284	.0275	.0267	.0259	.0249	.0236	.0224	.0216	.0209	.0203	.0198
7.0	.0415	.0385	.0365	.0348	.0324	.0307	.0292	.0282	.0272	.0265	.0257	.0247	.0234	.0222	.0214	.0207	.0201	.0196
8.0	.0412	.0382	.0362	.0346	.0322	.0304	.0290	.0280	.0270	.0263	.0255	.0245	.0232	.0221	.0213	.0205	.0200	.0195
10.0	.0408	.0378	.0358	.0342	.0319	.0300	.0288	.0277	.0267	.0260	.0253	.0243	.0229	.0218	.0211	.0203	.0198	.0193
12.0	.0403	.0375	.0355	.0339	.0316	.0298	.0285	.0275	.0265	.0258	.0251	.0241	.0228	.0217	.0209	.0201	.0196	.0191
16.0	.0398	.0370	.0350	.0334	.0312	.0294	.0281	.0271	.0261	.0254	.0247	.0237	.0224	.0214	.0206	.0198	.0193	.0188
20.0	.0395	.0366	.0347	.0331	.0308	.0292	.0278	.0268	.0259	.0252	.0245	.0235	.0222	.0212	.0204	.0197	.0191	.0186

[1] From "Hydraulic Tables," University of California Syllabus Series No. 246.

TABLE 16.—VALUES OF K_B FOR USE IN THE FORMULA $h_L = K_B \frac{V^2}{2g}$*

(90-deg. bends)

Radius of bend, ft.	Velocity V, ft./sec.												
	2	3	4	5	6	7	8	10	12	15	20	30	40
0.0	1.03	1.14	1.23	1.30	1.36	1.42	1.46	1.54	1.62	1.71	1.84	2.03	2.18
0.25	0.46	0.51	0.55	0.58	0.60	0.63	0.65	0.69	0.72	0.76	0.82	0.90	0.97
0.50	0.31	0.34	0.36	0.38	0.40	0.42	0.43	0.46	0.49	0.51	0.54	0.60	0.65
1.	0.21	0.23	0.25	0.26	0.28	0.29	0.30	0.31	0.33	0.35	0.37	0.41	0.44
2.	0.19	0.21	0.22	0.23	0.24	0.25	0.26	0.28	0.29	0.31	0.33	0.36	0.39
3.	0.18	0.20	0.22	0.23	0.24	0.25	0.26	0.27	0.29	0.30	0.33	0.36	0.39
4.	0.18	0.20	0.21	0.23	0.23	0.25	0.26	0.27	0.28	0.30	0.32	0.35	0.38
5.	0.18	0.20	0.21	0.22	0.23	0.24	0.25	0.27	0.28	0.29	0.32	0.35	0.38
6.	0.18	0.19	0.21	0.22	0.23	0.24	0.25	0.26	0.28	0.29	0.31	0.35	0.37
7.	0.19	0.21	0.22	0.23	0.24	0.25	0.26	0.28	0.29	0.31	0.33	0.36	0.39
8.	0.21	0.23	0.25	0.26	0.27	0.28	0.29	0.31	0.32	0.34	0.37	0.41	0.44
10.	0.26	0.29	0.31	0.32	0.34	0.35	0.36	0.38	0.40	0.42	0.46	0.50	0.54
15.	0.37	0.41	0.43	0.46	0.48	0.50	0.52	0.55	0.57	0.61	0.65	0.72	0.77
20.	0.45	0.51	0.54	0.57	0.60	0.62	0.64	0.68	0.72	0.75	0.81	0.90	0.97
25.	0.50	0.56	0.59	0.63	0.65	0.69	0.71	0.75	0.79	0.83	0.89	0.99	1.06

* From "Handbook of Hydraulics" by King, 2d ed.

TABLE 17.—LENGTHS OF STANDARD PIPE TO ALLOW FOR VARIOUS SCREW FITTINGS, FEET[1]

Nominal pipe size, in.	Actual inside diameter, in.	Gate valve	Long-sweep elbow or on run of standard tee	Medium-sweep elbow or on run of tee reduced in size ¼	Standard elbow or on run of tee reduced in size ½	Angle valve	Close-return bend	Tee through side outlet	Globe valve
½	0.622	0.335	0.442	0.56	0.89	1.20	1.34	1.78	2.68
¾	0.824	0 475	0.627	0.79	1.27	1.71	1.90	2.52	3.80
1	1.049	0.640	0.844	1.07	1.72	2.30	2.56	3.40	5.12
1¼	1.38	0.902	1 19	1.51	2.42	3.24	3.61	4.80	7.22
1½	1.61	1.09	1.43	1.83	2.92	3.92	4.36	5.79	8.72
2	2.06	1.49	1.96	2.50	3.99	5.36	5.96	7.92	11.92
2½	2.46	1.86	2.46	3.13	5.00	6.72	7.47	9.93	14.94
3	3.06	2 46	3 25	4.11	6.66	8.87	9.86	13.11	19.72
3½	3.54	2 92	3.80	4.91	7.84	10.53	11.70	15.56	23.40
4	4.026	3.44	4.53	5.77	9.22	12.37	13.70	18.28	27.50
4½	4.506	3.95	5 20	6.63	10.60	14.22	15.80	21.01	31.60
5	5.047	4 57	6.00	7.68	12.20	16.47	18.30	24.33	36.60
6	6.065	5.72	7.55	9.61	15.30	20.61	22 90	30.45	45.80
7	7.024	6.90	9.10	11.59	18.50	24.84	27 60	36.70	55.20
8	7.981	8.10	10.69	13.60	21.70	29.16	32.40	43.09	64.80
10	10.020	10.70	14.10	17.97	28.70	38.52	42.80	56.92	85.60
12	12.090	12.50	17.80	22.68	36.20	48 60	54.00	71.82	108.00

[1] After Dean E. Foster, *Trans. A.S.M.E.*, vol. 42, p. 647, 1920.

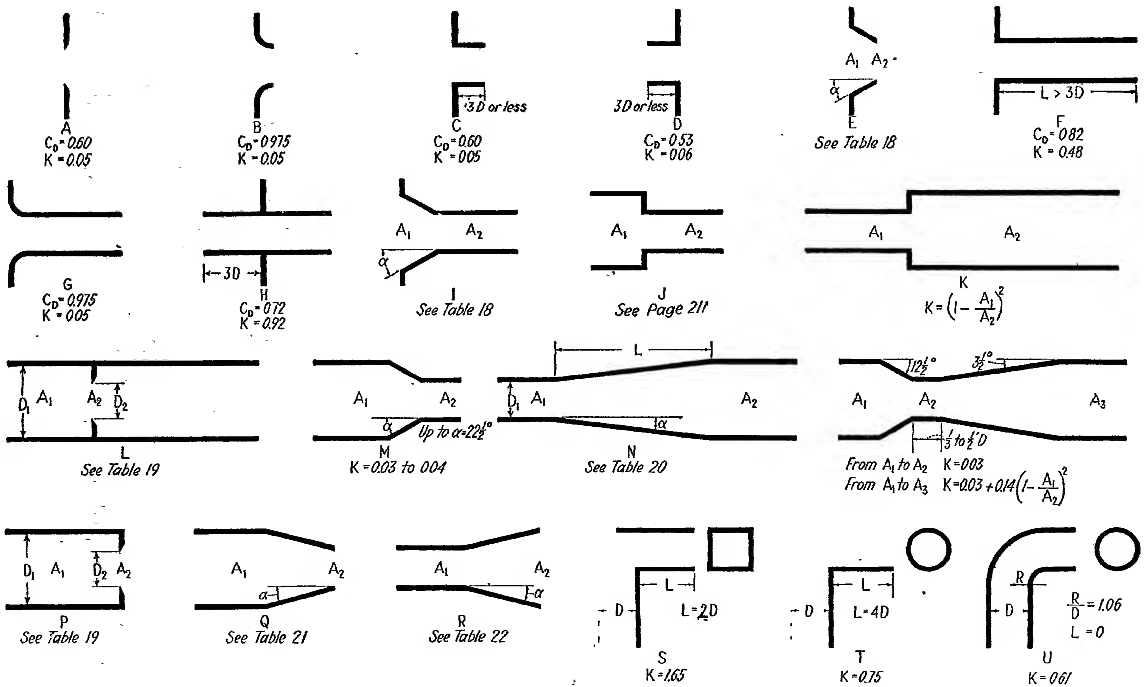

FIG. 166.—Values of C_D and K for some shapes. Flow is from left to right. $h_L = K\frac{V^2}{2g}$ $Q = C_D A\sqrt{2gh}$.

(Adapted from "Notes on the Flow of Gases," Frank Wills, P. G. & E. Co., San Francisco.)

TABLE 18.—COEFFICIENTS OF DISCHARGE AND ENERGY LOSS FOR CONVERGING SECTIONS[1]
(Fig. 166, E and I)

$\alpha°$	2	4	6	8	10	12	13	14	16	18	20	24	30	40	49
C_D*........	0.88	0 90	0.92	0.93	0.94	0.94	0.94	0.94	0.94	0.93	0.93	0.91	0.90	0.87	0.85
K..........	0.29	0.22	0.18	0.15	0.14	0.13	0.13	0.13	0.14	0.15	0.16	0.20	0.25	0.37	0.41

[1] From "Notes on the Flow of Gases," Frank Wills, Pacific Gas and Electric Company, San Francisco.

* For E only.

TABLE 19.—COEFFICIENTS OF DISCHARGE AND ENERGY LOSS FOR PIPELINE ORIFICES IN TERMS OF THE DIFFERENTIAL $h_1 - h_3$*
(Fig. 166, L and P)

D_2/D_1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
C_D..........................	0.605	0.609	0.616	0.630	0.654	0.701	0.788
K_{1-3}........................	0.12	0.11	0.09	0.04	0.06	0.05	0.07
K_{1-4}........................	0.046	0.042	0.035	0.028	0.025	0.022	0.036

* From "Notes on the Flow of Gases," Frank Wills, Pacific Gas and Electric Company, San Francisco.

See also pp. 165 and 171.

TABLE 20.—COEFFICIENTS OF ENERGY LOSS FOR GRADUAL EXPANSION IN PIPELINES[1]
(Fig. 166, N)
Values of k in $K = k[1 - (A_1/A_2)^2]$

$\alpha°$	L/D_1							
	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5	$7\frac{1}{2}$
2	0.114	0.191	0.267	0.268	0.298	0.295	0.283	0.246
3	0.161	0.243	0.294	0.306	0.316	0.297	0.280	0.263
4	0.195	0.286	0.325	0.329	0.332	0.306	0.301	
5	0.227	0.319	0.350	0.352	0.350	0.334		
6	0.256	0.346	0.371	0.377	0.377	0.365		
7	0.283	0.373	0.393	0.400	0.400			
8	0.305	0.396	0.414	0.424				
9	0.324	0.415	0.434	0.445				
10	0.352	0.435	0.457	0.468				
11	0.360	0.457	0.478	0.494				
12	0.394	0.469	0.500					
13	0.415	0.492	0.518					
14	0.440	0.511	0.535					
15	0.463	0.530						
20	0.578	0.646						
25	0.690							
30	0.818							

[1] From "Fan Engineering," Madison and Carrier.
See also Fig. 104.

TABLE 21.—ENERGY-LOSS COEFFICIENTS FOR CONVERGING NOZZLE AT END OF PIPE[1]
(Fig. 166, Q)
Values of k in $K = k[1 - (A_1/A_2)^2]$

$\alpha°$	10	12	14	16	18	20
k	0.06	0.07	0.09	0.11	0.13	0.15

[1] From "Notes on the Flow of Gases," Frank Wills, Pacific Gas and Electric Company, San Francisco.
See also p. 213.

TABLE 22.—ENERGY-LOSS COEFFICIENTS FOR DIVERGING NOZZLE AT END OF PIPE[1]

(Fig. 166, R)

Values of k in $K = k[1 - (A_1/A_2)^2]$

$\alpha°$.	1	2	3	4	5	6	7	8	9	10	15	20	25	30
k. . .	0.10	0.15	0 20	0.24	0.28	0.31	0.35	0.38	0.41	0.44	0.59	0.73	0.86	1.00

[1] From "Notes on the Flow of Gases," Frank Wills, Pacific Gas and Electric Company, San Francisco.

See also Fig. 104.

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